MATH 115 – FINAL EXAM

December 17, 2007

NAME: SOLUTIONS

INSTRUCTOR: ______________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	7	
3	7	
4	12	
5	10	
6	10	
7	16	
8	14	
9	12	
TOTAL	100	

1. (2 points each) Indicate if the following statements are **always** true, T. If a statement is never true or only sometimes true indicate it is false, F.

(a) If $f(a) < f(x)$ for all x in $[a, b]$, then $f(x)$ is increasing on $[a, b]$.

______________________________ **T** $\boxed{\textbf{F}}$

(b) Every function has a global maximum on a closed interval $[a, b]$.

______________________________ **T** $\boxed{\textbf{F}}$

(c) $\displaystyle\int_a^b g(x)^2 dx = \left(\int_a^b g(x)dx\right)^2$

______________________________ **T** $\boxed{\textbf{F}}$

(d) If $f'(x) = g'(x)$ for all x and $f(0) = g(0)$, then $f(x) = g(x)$.

______________________________ $\boxed{\textbf{T}}$ **F**

(e) A continuous, differentiable function, g, with three critical points in the range $0 \leq t \leq 10$ has at least four changes in concavity.

______________________________ **T** $\boxed{\textbf{F}}$

(f) The function $p(t) = \dfrac{c}{3}t^3 + Ac$ (A and c constants) is an antiderivative of $q(t) = ct^2$.

______________________________ $\boxed{\textbf{T}}$ **F**

2. (7 points) Use a Riemann Sum with 4 equal subdivisions to find a ***lower*** estimate for

$$\int_0^2 e^x + 1\,dx.$$

Clearly indicate whether you are using a left-hand sum or a right-hand sum, and show all intermediate calculations. Show your answer to three decimal places (or in exact form).

The function is increasing, therefore the Left Sum is the lower sum.

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
e^x+1	2	$e^{1/2}+1$	$e+1$	$e^{3/2}+1$	e^2+1
e^x+1	2	2.6487	3.7183	5.4817	8.3891

Left Sum

$$LHS_{(4)} = (0.5)(2) + (0.5)(e^{1/2}+1) + (0.5)(e+1) + (0.5)(e^{3/2}+1) = 6.9243$$

3. (7 points) Let $f(x) = \cos(x) + bx$ and $g(x) = x^2 - x$. Find the value of b such that $f(x) > g(x)$ on $[0, 1]$ and the area between the curves from $x = 0$ to $x = 1$ is equal to 1.

$$1 = \int_0^1 \cos x + bx - (x^2 - x)dx$$

$$1 = \sin(1) - \frac{1}{3} + \frac{b+1}{2}$$

$$b = \frac{5}{3} - 2\sin(1) = -0.0162753$$

[Note that with this value of b, $f(x) > g(x)$ on $[0, 1]$–but, with the set-up of the problem as indicated above, we are assuming that $f(x) > g(x)$ on the interval.]

4. (12 points) Suppose that f, g and h are all continuous and differentiable functions such that:

- f is an odd function
- $\int_0^3 f(t)dt = 3$
- $g(t) = t^2 + 2$
- $h(t) = g'(t-1)$

Evaluate the following, where possible. If evaluation is not possible, simply state "insufficient information."

(a) $\int_{a+3}^{a+3} f(t)\,dt = 0$

(b) $\int_{-10}^{10} f(t)\,dt = 0$ (since f is odd)

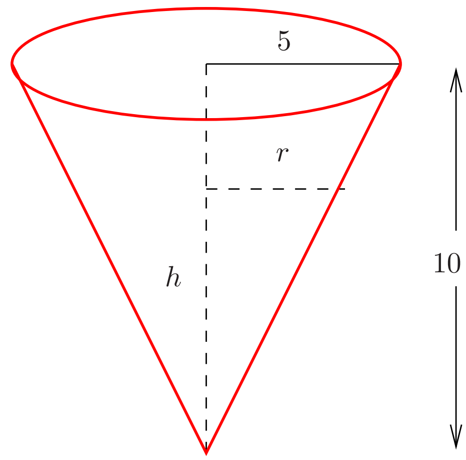
(c) The average value of g on the interval $[-2, 2]$

$$\frac{1}{4}\int_{-2}^{2}(t^2+2)dt = \frac{1}{4}\left(\frac{t^3}{3}+2t\right)\Big|_{-2}^{2} = \frac{10}{3}$$

(d) $\int_{-3}^{0} f(t)\,dt = -3$

(e) $\int_{-1}^{1} h(t)\,dt = g(0) - g(-2) = 2 - (4+2) = -4$

5. (10 points) Madam Whippy's ice-cream store has a vending machine that pumps out manilla vanilla at a constant rate of 2 cm^3 per second. If you are collecting the ice-cream in a cone of maximum radius 5 cm and height 10 cm, how fast is the radius of the surface of the ice-cream changing when the height of ice cream in the cone is 6 cm? Assume that the ice cream is soft-serve and fills the cone with a flat surface.
[You may need: Volume of a cone of height h and radius r is given as $V = \frac{1}{3}\pi\, r^2\, h$.]



Using similar triangles,

$$\frac{r}{h} = \frac{5}{10} \;\rightarrow\; h = 2r$$

Thus $V = \frac{2}{3}\pi\, r^3$ and

$$\frac{dV}{dt} = 2\,\pi r^2\,\frac{dr}{dt}$$

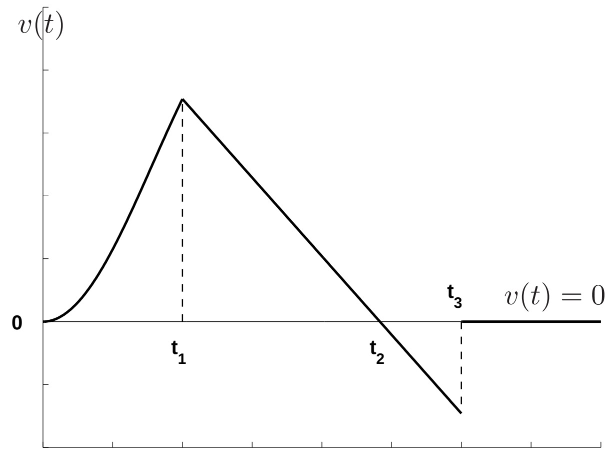
and since we are given $\frac{dV}{dt} = 2$, we have

$$2 = 2\,\pi r^2\,\frac{dr}{dt} \;\rightarrow\; \frac{dr}{dt} = \frac{1}{\pi r^2}$$

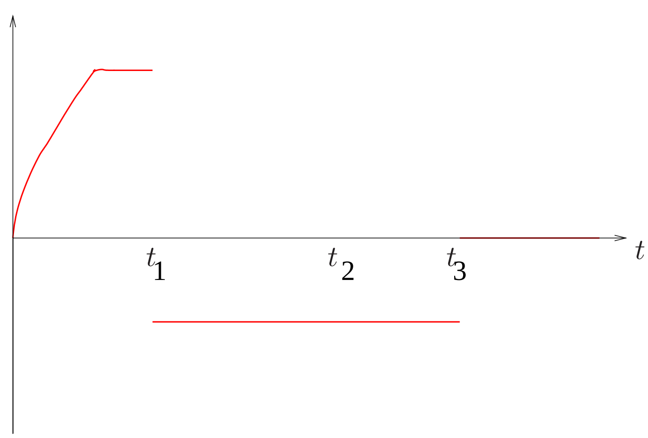
which at $h = 6$ and $r = 3$ gives

$$\frac{dr}{dt} = \frac{1}{9\pi} \quad \text{cm/s.}$$

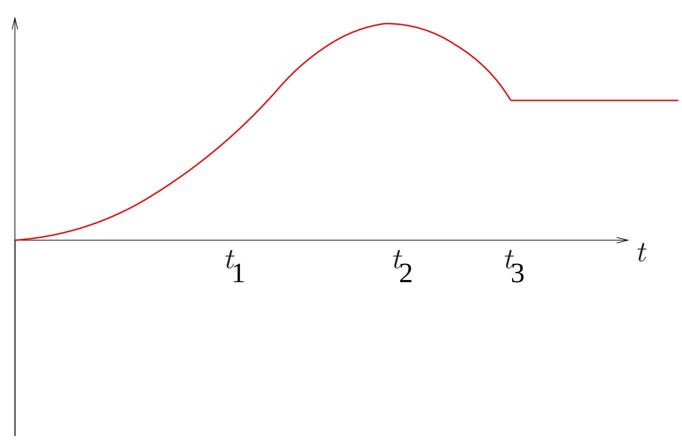
6. (10 points) A budding rocket scientist, Seema, has launched her model rocket from the ground at time $t = 0$. The velocity profile for Seema's rocket is given in the graph below. (Note: Since the vertical scale is not given, we are interested in the "shapes" and general behavior on the graphs below.)



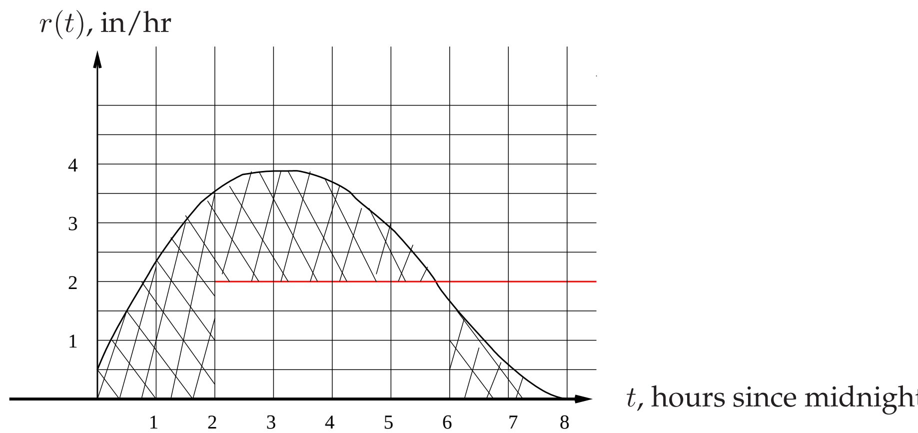
(a) Sketch a graph of the acceleration of the rocket as a function of time on the axes below.



(b) Sketch a graph of the height of the rocket as a function of time.



7. (16 points) One winter a huge storm hit the Ann Arbor area hours earlier than expected. Snow began falling at midnight and quickly began to accumulate. Although emergency conditions were put into place immediately, snow removal teams were unable to get onto the roads until 2:00 a.m. Particular attention was given along I-94, which needed to be cleared for heavy commuter traffic. The rate of snowfall along that highway strip is given by the graph below, where $r(t)$ is in inches per hour, and t is hours past midnight. Assume that once the removal starts, plows can remove the snow at an average rate of 2 in/hour. Use the graph to find or to estimate the following:



[Answers below are not unique.]

(a) The amount of snowfall that had accumulated prior to the time the plows got on the road at 2:00 a.m.

Note, each rectangle is 1/2 by 1 so each rectangle has area 1/2. Using rectangles to approximate the area under r from $t = 0$ to 2, gives approximately 9 rectangles= 1/2(9)=4.5 inches.

(b) The rate at which the snow level is changing at 3:00 a.m.

At 3:00 a.m., the snow is falling at approximately 3.75 in/hour and is being plowed at 2 in/hour. Thus, it is is accumulating at approximately 1.75 inches per hour.

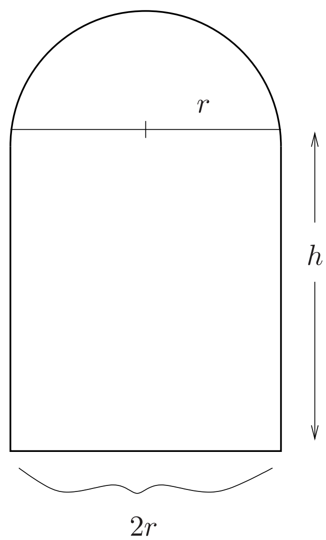
(c) The time when the level of accumulated snow is maximum.

The level will be at the maximum when the plowing rate overtakes the falling rate–or at approximaely 5:45 a.m.

(d) The time when the highway was cleared of snow, assuming that the snowfall stopped at 8:00 a.m.

We are interested in when the area under r is cleared–which is equivalent to considering how long it would take to clear the snowfall represented by the shaded area in the figure above. We have already found the area from $t = 0$ to $t = 2$ to represent approximately 4.5 inches, and there are approximately 13 rectangles in the shaded regions between $t = 2$ and $t = 8$ when the snow stops. Thus, we need to clear $4.5 + 0.5(13) = 11$ inches of snow in excess of what has been cleared by the plows at 5:45 a.m. At a rate of 2 inches per hour, it will take approximately 5.5 hours after the plowing rate exceeds the falling rate– or until around 11:15 a.m.

8. (14 points) Adrian is decorating her apartment for the holidays. She plans to outline a fake window with fancy lights. The window is to be in the shape of a rectangle topped by a semicircle (see Figure below). The cost of the fancy lights for the circular portion of the window is \$25 per meter, and the cost of the regular lights for the straight edging is \$10 per meter. If the "window" has a total area of 1 m^2, what are the dimensions that will minimize the cost of the lighted perimeter?



If r is the radius of the semicircle and h is the height of the straight side, then the area is

$$A = 2rh + \frac{\pi r^2}{2} = 1,$$

so

$$h = \frac{(1 - \frac{\pi r^2}{2})}{2r} = \frac{1}{2r} - \frac{\pi r}{4}.$$

Thus, the cost is

$$C = 25\pi r + 10\left(2r + 2h\right),$$

which gives

$$C = 25\pi r + 10\left(2r + 2\left(\frac{1}{2r} - \frac{\pi r}{4}\right)\right) = 25\pi r + 20r + \frac{10}{r} - 5\pi r.$$

Then

$$C' = 25\pi + 20 - \frac{10}{r^2} - 5\pi.$$

$C'(r) = 0$ if $\frac{10}{r^2} = 20\pi + 20$, so

$$r^2 = \frac{1}{2\pi = 2} \approx 0.3475 \quad \text{meters.}$$

Checking for minimum, we have

$$C''(r) = \frac{10}{r^3}$$

which is greater than zero for all $r > 0$, so $r = 0.3475$ is a local minimum. Since this is the only critical point, it is the global minimum.

When $r = 0,3475, h = 1.166$, so the dimensions are base $= 0.695$ meters, and height $= 1.166$ meters.

9. (12 points) ... and we end where we began Calculus—with derivatives (mostly).

Given:

- $r(2) = 4$
- $s(2) = 1$
- $s(4) = 5$
- $r'(2) = -1$
- $s'(2) = 3$
- $s'(4) = 3$

Compute the following or state what additional information you would need in order to do so.

(a) $H'(2)$ if $H(x) = \sqrt{r(x)}$

$$H'(x) = \frac{1}{2}\left(r(x)^{-1/2}\right)\ r'(x),$$
$$H'(2) = -\frac{1}{4}$$

(b) $H'(2)$ if $H(x) = r(s(x))$

$$H'(x) = r'(s(x))s'(x)$$
$$H'2) = (r'(1))(3) \quad \text{but } r'(1) \text{ is missing}$$

(c) $H'(2)$ if $H(x) = xe^{s(x)}$

$$H'(x) = e^{s(x)} + xe^{s(x)}s'(x)$$
$$H'(2) = 7e$$

(d) $\displaystyle\int_2^4 s'(x)dx$

$$\int_2^4 s'(x)dx = s(4) - s(2) = 4$$

Happy winter break and wonderful holidays to all!!