MATH 115 –FIRST MIDTERM

October 7, 2008

NAME: ______________SOLUTIONS______________

INSTRUCTOR: ______________ SECTION NUMBER: ______________

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1. **Do not open this exam until you are told to begin.**
2. This exam has 10 pages including this cover. There are **??** questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	14	
2	15	
3	12	
4	10	
5	12	
6	10	
7	9	
8	10	
9	8	
TOTAL	100	

1. In 1999, a population of deer (a type of large animal) was set free on a previously uninhabited island in Lake Superior, in an attempt to establish a permanent population of deer on the island. The population of deer grew over time. Population measurements were made each year, as shown in the following table:

Year	1999	2000	2001	2002
Population	20	23	27	31

Let $P(t)$ be a function that gives the population of deer on the island as a function of time, t, measured in years since 1999.

(a) (2 points) In the context of this problem, give a practical interpretation for $P(40)$.

$P(40)$ represents the deer population on the island in the year 2039.

(b) (2 points) In the context of this problem, give a practical interpretation for $P^{-1}(40)$.

$P^{-1}(40)$ is the number of years after 1999 for which the deer population on the island was 40.

(c) (3 points) Assume that the deer population at time t is represented by an exponential function $P(t) = P_0 a^t$. Find P_0 and a, and express your answer as a function.

From the table we know that at $t = 0$ the population is 20, and also that at $t = 1$ the population is 23. Thus, using the point (0,20) gives the initial value $P_0 = 20$, which gives the equation as $P(t) = 20a^t$. Now, using the second point, (1,23), we can solve for a via $23 = 20a$. Using this, we arrive at the final equation, $P(t) = 20(1.15)^t$.

(d) (2 points) According to your answer to part (c) what is the annual percent growth rate of the deer population?

The annual percent growth rate is the growth factor minus one, expressed in percent form, $(1.15 - 1) \times 100 = 15\%$.

(e) (3 points) Use the table above to estimate $(P^{-1})'(27)$. Do not assume that the deer population is modeled by the formula from part (c).

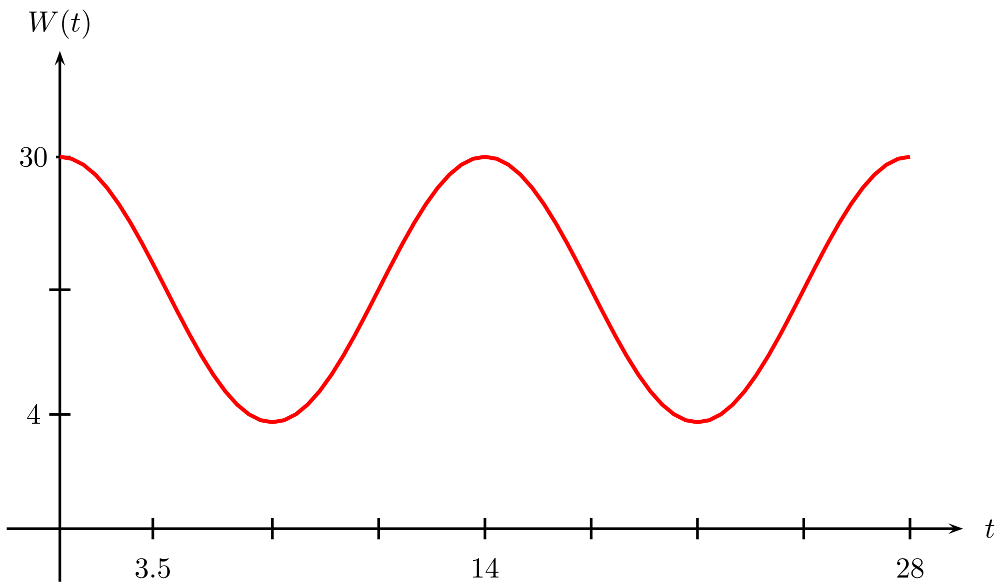
First, we need some values for P^{-1}. From the table we see that $P^{-1}(27) = 2$ and $P^{-1}(31) = 3$. Thus, we can estimate the derivative as $(P^{-1})'(27) \approx \frac{3-2}{31-27} = 0.25$.

(f) (2 points) Give a practical interpretation in the context of this problem for $(P^{-1})'(27)$.

When there were 27 deer on the island, it took approximately 1/4 of a year, or 3 months, for the population to reach 28.

2. Before Hurricane Ike hit Galveston, TX, on September 10, 2008, there were reports of waves up to 26 ft. tall, with offshore buoys recording 14 seconds between waves. The water level along the seawall was already 4 feet above normal sea level, and the city of Galveston was concerned that the storm surge might overwhelm the seawall. Let $W(t)$ be the trigonometric function giving the height of the water above sea level at time t, where t is measured in seconds since one of the waves crashed against the seawall. Assume the minimum height is 4 feet and the waves surge from and return to that level.

   (a) (5 points) On the axes below, sketch two periods of $W(t)$. Be sure to include important axes markings and labels.



   (b) (7 points) Find a formula for $W(t)$.

   We can start by finding the midline (the average of the high and low of the wave), which is $(30 + 4)/2 = 17$. Next, the amplitude is 30 minus the midline, which is 13. Now, since the period is $T = 14$ seconds, the coefficient of t is $\frac{2\pi}{T} = \frac{\pi}{7}$. Lastly, since the wave starts at its high of 30 ft., this can most easily be modeled by a cosine graph. Thus, our final equation is $W(t) = 13\cos\left(\frac{\pi}{7}t\right) + 17$.

   (c) (3 points) For what t value, $0 \leq t \leq 14$, is the wave height decreasing the fastest?

   The wave height is decreasing the fastest when it crosses its midline downwards. This happens only once in the given interval, at $t = 3.5$.

3. (3 pts each–no partial credit) The following problems are to be considered independent of each other. For each problem, circle **all** the statements that are correct.

   (a) Let $C(r)$ represent the total cost of paying off a car loan borrowed at an interest rate of $r\%$ per year. Then:

   - The units of $C'(r)$ are \$/year.
   - The expression $C'(5) = A$ (with units) represents the rate of change of the total cost of the car loan.
   - The expression $C'(5) = A$ (with units) indicates that if the interest rate increases from 5% to 6%, the total cost of the loan would be approximately $C(5) + A$.
   - The expression $C'(5)$ (with units) indicates that if the interest rate increases by 5%, then the total cost of the loan increases by about $C'(5)$.
   - The expression $C'(5)$ (with units) indicates that if the interest rate increases from 5% to 6%, the total cost of the loan increases by about $C'(5)$.
   - The sign of $C'(5)$ cannot be determined from the context of the information given.

   (b) If the figure below shows position as a function of time for two sprinters running in parallel lanes, then:

   - At time A, both sprinters have the same velocity.
   - Both sprinters continually increase their velocity.
   - Both sprinters run at the same velocity at some time before A.
   - At some time before A, both sprinters have the same acceleration.

   (c) Let f and g be differentiable functions. Assume f is an even function and g is an odd function. Then:

   - g' is an even function.
   - the composition, $f(g(x))$, is an odd function.
   - $h(x) = f(x)g(x)$ is an odd function.

   (d) Suppose that $f''(x) > 0$ everywhere. Then:

   - $f'(x)$ is increasing.
   - $f(b) > f(a)$ whenever $a < b$.
   - $f'(x) < 0$.

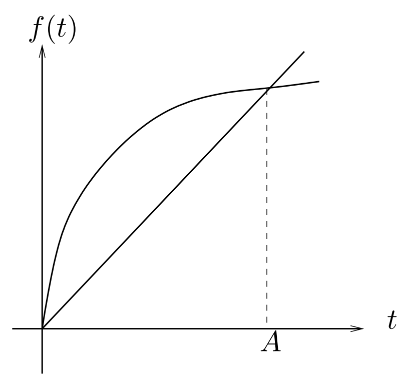


Figure for part (b)

4. The speed of sound, $v(T)$ (in miles per hour), at an ambient temperature, T (in degrees Farenheit), is given by:

$$v(T) = 740 + 0.4T.$$

Objects which travel faster than the speed of sound create *sonic booms*. However, the ambient temperature T in the Troposphere also decreases with height h (in miles) from Earth's surface according to the equation

$$T(h) = -26h + T_0,$$

where T_0 is the temperature at the surface.

(a) (3 points) Find a formula which will give the speed of sound S as a function of height h, assuming the surface temperature is $68°$F.

We are looking for the composite function $S(h) = v(T(h)) = 740 + 0.4(-26h + 68) = 767.2 - 10.4h$.

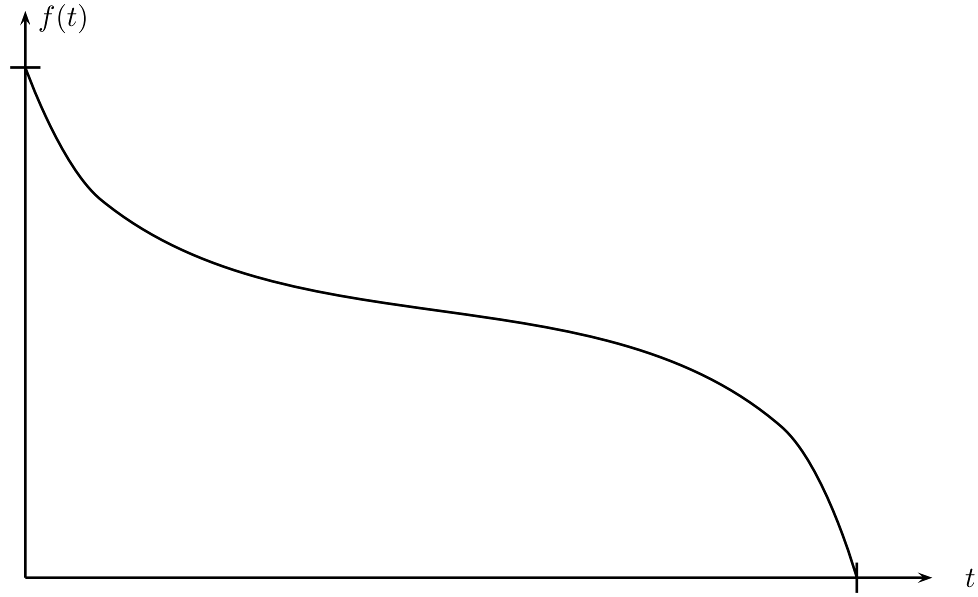
(b) (4 points) Find $S'(1)$ and interpret the meaning of $S'(1)$ in the context of this problem.

Since S is linear, the derivative at any point is the same as the line's slope. Thus, $S'(1) = -10.4\frac{mi/hr}{mi}$. Moreover, we can interpret this as telling us that the speed of sound 2 miles above the Earth's surface is approximately 10.4 mi/hr less than the speed of sound 1 mile above the Earth's surface.

(c) (3 points) While on a flight from Ann Arbor to Chicago on a beautiful $68°$ day, the pilot's instruments measure the outside temperature to be $0°$. What is the plane's altitude, and how fast would the pilot need to fly at this altitude to create a sonic boom?

We first need to solve for the plane's altitude. We can do this by solving the equation $0 = -26h + 68$, which gives $h \approx 2.61$ miles. Then, we can find the speed of sound at this altitude via $S(2.61) \approx 740$ mi/hr. Thus, at this altitude and ambient temperature the pilot would need to fly faster than this speed in order to create a sonic boom.

5. The graph below shows an approximation to the stock price, $P = f(t)$ in dollars, of Lehman Brothers Inc. (LEH) with t measured in months since the stock's highest point in February 2007 to the company's ultimate bankruptcy in September 2008 (t=19).



(a) (2 points) Explain why f is invertible on the indicated domain.

f is invertible on its domain because for each output there is a unique input. Graphically, this is best seen by the fact that the graph passes the horizontal line test.

(b) (3 points) Interpret, in the context of this problem, $f^{-1}(5)$.

$f^{-1}(5)$ is the number of months (since Feb 2007) it took for LEH's stock price to reach \$5.

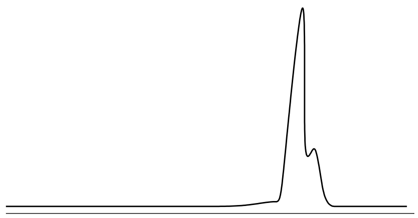
(c) (4 points) If $\frac{dP}{dt}|_{t=16} = -5$ and $f(16) = 25$, find an equation of the line tangent to the curve at $t = 16$.

Since we are given the slope and a point, we can use the point-slope equation. Thus, our tangent line equation is $y - 25 = -5(t - 16)$, or equivalently $y = -5t + 105$.

(d) (3 points) Using part (c), what month would your tangent line have predicted LEH's stock price would reach zero?

We need to know what t value yields $y = 0$. From part (c), we get the equation $-25 = -5t + 80$, which has the solution $t = 21$. Now, since t was measured in months since Feb 2007, $t = 21$ corresponds to November 2008 (Lehman actually filed for bankruptcy in September of 2008).

6. (2 points each) Google Trends is an online website which tracks how frequently certain search strings are entered into Google. Entering the term "pumpkin" produces a graph similar to the graph below.



Google Trend for "pumpkin"

Not surprisingly, this graph is basically periodic over a 12-month period. Suppose we call this function $P(t)$, where the horizontal axis represents time, t in months since December (so $t = 1$ is January of any given year). The values of $P(t)$ represent how often a term is searched for, relative to the total number of searches. The spike in the pumpkin graph, again not surprisingly, comes around $t = 10$ each year. (We figure the second, smaller spike represents queries about what to do with rotting pumpkins....) Other trends are seasonal as well–*e.g.*, "summer camps." On the other hand, some searches have a quick peak and die forever (or at least for longer than a year)–*e.g.*, "Vice Presidential debates."

Assume that the peak in the graph above occurs at the point (10, 100). Use this information to determine the coordinates of the peak for the following searches that have similar patterns but peak at different points. On each line below, give the coordinates of the peak in the new function, given that function's relationship to the function P.

(a) The peak for the function C if $C(t) = 10P(t)$. (10, 1,000)

(b) The peak for the function K if $K(t) = P(t+2)$. (8, 100)

(c) The peak for the function G if $G(t) = P(t)+2$. (10, 102)

(d) The peak for the function H if $H(t) = 3P(t-5)+1$. (15, 301)

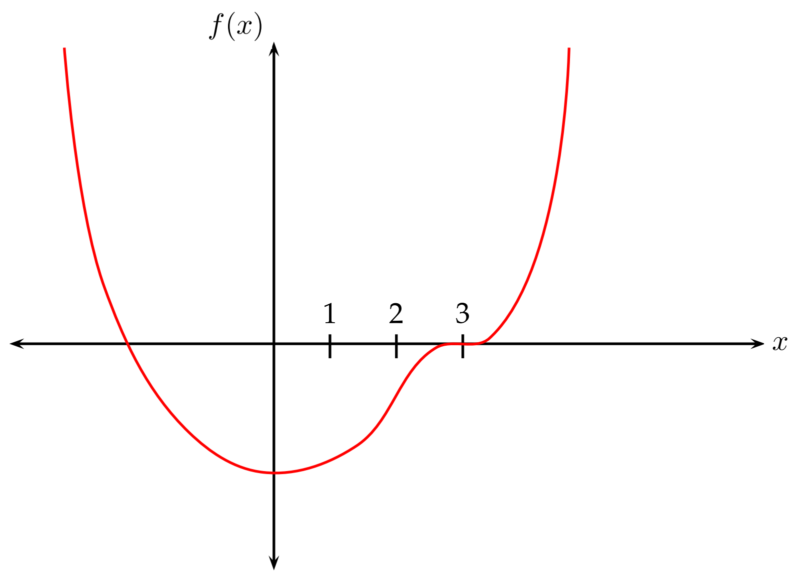
(e) In the context of this problem, does $P(-10)$ make sense? If so, what would that mean? If not, explain why not.

Yes, $P(-10)$ represents the same scenario in the previous February (recall that $P(-t)$ represents the graph of $P(t)$ reflected about the vertical axis).

7. (9 points) A continuous, differentiable function defined for all x has all of the following properties:

- $f'(x) = 0$ at $x = 0$ and $x = 3$
- $f(3) = 0$
- $f'(-1) = -2$
- f' is increasing for $x < 2$
- $f' \geq 0$ for $x > 0$
- $\lim_{x \to -\infty} f(x) = \infty$

(a) (3 points) Sketch a possible graph of f. (One possible graph is shown below.)



(b) (2 points) How many zeroes does f have? Explain your reasoning.

red$f$ will have infinitely many zeroes if $f' = 0$ for $x \geq 3$, and 2 otherwise. (Either answer is acceptable.)

(c) (2 points) What can you say about the location of the zeroes? Explain your reasoning.

From the given data, we can see that there is one zero for $x < 0$, one zero at $x = 3$, and if $f' = 0$ for $x \geq 3$ we have infinitely many zeroes.

(d) (2 points) Is it possible that $f'(-2) = -1$? Explain your reasoning.

No, since f' is increasing (or equivalently, f is concave up), $f'(-2) < f'(-1)$–(*i.e.*, $f'(-2)$ is more negative).

8. The function $L(x) = \dfrac{1}{\ln(x)}$ is differentiable over its domain.

(a) (2 points) What is the domain of L?

The domain of L is $x > 0$, excluding $x = 1$.

(b) (4 points) Write the formula for the derivative of L at $x = a$ using the *limit definition* of the derivative.

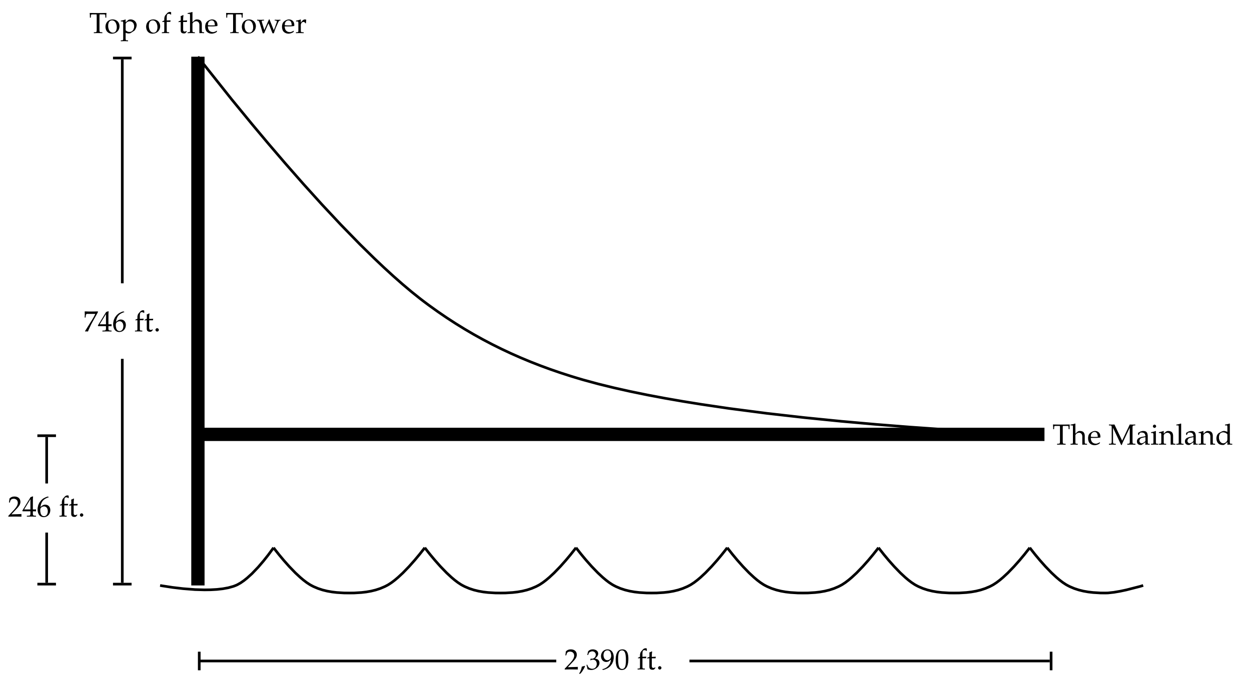
$$L'(a) = \lim_{h \to 0} \frac{\frac{1}{\ln\,(a+h)} - \frac{1}{\ln a}}{h}$$

.

(c) (4 points) Given $\frac{dL}{dx}|_{x=2} = -1.0407$ and $\frac{dL}{dx}|_{x=2.5} = -.4764$ and given that the derivative is *monotonic* (meaning the derivative does not change behavior from decreasing to increasing or vice versa) for all $x > 1$, what does this information tell you about the graph of L for x near 2? Explain your reasoning using words and symbols (*i.e.*, **not** by drawing a graph).

The graph of L near $x = 2$ must be concave up, since the derivative is increasing and monotonic. The function is decreasing, since the derivative is negative. (Also, we know that the graph of ln x is increasing, and L is defined as the reciprocal of ln x, so L is decreasing).

9. San Francisco's famous Golden Gate bridge has two towers which stand 746 ft. above the water, while the bridge itself is only 246 ft. above the water. The last leg of the bridge, which connects to Marin County, is 2,390 ft. long. The suspension cables connecting the top of the tower to the mainland can be modeled by an exponential function. Let $H(x)$ be the function describing the height above the water of the suspension cable as a function of x, the horizontal distance from the tower.



(a) (4 points) Find a formula for $H(x)$.

We are looking for a formula of the form $H(x) = H_0 a^x$. We can use the given information to extract the two points which we'll use to find our exponential function: (0,746) and (2390,246). The first of these points gives use the initial value, and from the second we can form the equation $246 = 746a^{2390}$, which can be solved for a. Thus, our final equation is $H(x) = 746(0.9995)^x$, or $H(x) = 746e^{-0.000464x}$.

(b) (4 points) The engineers determined that some repairs are necessary to the suspension cables. They climb up the tower to 400 ft above the bridge, and they need to lay a horizontal walking board between the tower and the suspension cable. How long does the walking board need to be to reach the cable?

We are looking for an x-value, given that the height up the tower is $246 + 400 = 646$. Thus, we must solve the equation $646 = 746(0.9995)^x$. Solving this equation yields about 287.78 ft, or, if using the second form, $x \approx 310$ ft. (Note: the variance in answers is due to round-off in the representations. Either form is accepted.)