MATH 115 – SECOND MIDTERM

November 18, 2008

NAME: ______________SOLUTIONS______________________

INSTRUCTOR: ____________________ SECTION NUMBER: ____________

1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam. If you need extra room, you may use the back of a page but be *sure* to clearly indicate and label your work.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	10	
2	12	
3	18	
4	8	
5	6	
6	16	
7	16	
8	14	
TOTAL	100	

1. For the following questions select true if the statement is *always* true, and false otherwise. Each question is worth 1 point.

   (a) If f is differentiable and $f'(p) = 0$ or $f'(p)$ is undefined, then $f(p)$ is either a local maximum or a local minimum.

   True $\boxed{\text{False}}$

   (b) For f a twice differentiable function, if f' is increasing, then f is concave up and increasing.

   True $\boxed{\text{False}}$

   (c) The global maximum of $f(x) = x^2$ on every closed interval is at one of the endpoints of the interval.

   $\boxed{\text{True}}$ False

   (d) If $f(x)$ has an inverse function $g(x)$, then $g'(x) = 1/f'(x)$.

   True $\boxed{\text{False}}$

   (e) If a function is periodic with period c, then so is its derivative.

   $\boxed{\text{True}}$ False

   (f) If $C(q)$ represents the cost of producing a quantity q of goods, then $C'(0)$ represents the fixed costs.

   True $\boxed{\text{False}}$

   (g) If a differentiable function $f(x)$ has a global maximum on the interval $0 \le x \le 10$ at $x = 0$, then $f'(x) \le 0$ for $0 \le x \le 10$.

   True $\boxed{\text{False}}$

   (h) If $f(x)$ is differentiable and concave up, then $f'(a) < \frac{f(b)-f(a)}{b-a}$ for $a < b$.

   $\boxed{\text{True}}$ False

   (i) If you zoom in with your calculator on the graph of $y = f(x)$ in a small interval around $x = 10$ and see a straight line, then the slope of that line equals the derivative $f'(10)$.

   $\boxed{\text{True}}$ False

   (j) If $f'(x) \ge 0$ for all x, then $f(a) \le f(b)$ whenever $a \le b$.

   $\boxed{\text{True}}$ False

2. In 1956, Marion Hubbert began a series of papers predicting that the United States' oil production would peak and then decline. Although he was criticized at the time, Hubbert's prediction was remarkably accurate. He modeled the annual oil production $P(t)$, in billions of barrels of oil, over time t, in years, as the *derivative* of the *logistic function* $Q(t)$ given below—*i.e.*, $Q'(t) = P(t)$. The function P is measured in years since the middle of 1910.

   The function $Q(t)$ is given by

$$Q(t) = \frac{Q_0}{1 + ae^{-bt}}\text{ , where } a, b, Q_0 > 0. \tag{1}$$

   For your convenience, the first and second derivatives of $Q(t)$ are given as well:

$$Q'(t) = -\frac{Q_0}{(1 + ae^{-bt})^2}\left(-abe^{-bt}\right) = \frac{abQ_0e^{-bt}}{(1 + ae^{-bt})^2},$$

   and

$$Q''(t) = \frac{ab^2Q_0e^{-bt}}{(1 + ae^{-bt})^3}\left[ae^{-bt} - 1\right].$$

   (a) (2 points) Interpret, in the context of this problem, $P'(56)$.

   $P'(56)$ is approximately the number of billions of barrels by which the United States' annual oil production increased from the middle of 1966 to the middle of 1967. (If $P'(56)$ is negative, then this represents a decrease in production during that time period.)

   (b) (6 points) Determine the year of maximum annual production t_{max}. Your answer may involve all or some of the constants a, b, Q_0.

   We have that $P'(t) = Q''(t) = \frac{ab^2Q_0e^{-bt}}{(1+ae^{-bt})^3}\left[ae^{-bt} - 1\right]$. The factor preceding the bracketed term is positive for all t. The bracketed term changes sign once, at $t = (1/b)\ln(a)$; so this is the only critical point of $P(t)$. The global maximimum occurs at this point, because $P'(t)$ is positive before that point and negative afterward. Thus, $t_{max} = \frac{1}{b}\ln a$.

   (c) (2 points) Find the maximum annual production $P(t_{max})$. Again, your answer may involve all or some of the constants a, b, Q_0.

   Using t_{max} from part (b), we get $P(t_{max}) = \frac{1}{4}bQ_0$.

   (d) (2 points) In his 1962 paper, Hubbert studied the available data on oil production to date and concluded that $a = 46.8$, $b = 0.0687$, and $Q_0 = 170$ Bb (billion barrels). Using your results from part (b), when would Hubbert's curve predict the peak in US oil production? (The actual peak occurred in 1964.)

   Using the results of part (b), we get $t_{max} = \frac{1}{0.0687}\ln(46.8) \approx 55.98$, which corresponds roughly to the middle of 1966.

3. Use the information below to find an equation that best models the situation and most accurately fits the given data.

   (a) i. (2 points) Suppose a pair of shoes at DSW costs \$50 after a 10% discount. Find a formula for $P(n)$, the price of the shoes after n discounts of 10%, where $n \geq 0$.

   Since each successive discount makes the price 90% of the previous price, the function $P(n)$ is exponential. Moreover, we are given that $P(1) = 50$. Thus, using $P(n) = P_0 a^n$, we know that $a = 0.90$, and that $50 = P_0(0.90)$. From here we get $P_0 = (50/0.90)$, and arrive at $P(n) = (\frac{50}{0.90})(0.90)^n$.

   ii. (4 points) Find and interpret $P'(4)$ in the context of this problem.

   $P'(n) = \ln(0.90)(\frac{50}{0.90})(0.90)^n$, from which we find that $P'(4) \approx -3.84$ \$/discount. The units give us a hint for the interpretation: After the 4th discount, the 5th discount will lower the price of the shoes by approximately an additional \$3.84.

   (b) (6 points) Michigan's population (in millions) for the last three years as measured by the U.S. Census Bureau is given below.

Year	2005	2006	2007
Population	10.108	10.102	10.071

   Find a formula to approximate the population of Michigan, $P(t)$, with t in years since 2005. Using this information, approximate the population of Michigan in 2008. Show your work.

   We may approximate $P(t)$ by a linear function or an exponential function—neither exactly fits the data. Multiple answers were accepted here. [Note: the grading on this problem was based on the reasonableness of your answer to the first part of the question (and reasoning)– and, if that reasoning is sound, the second part of the answer was based on the first answer– units must be included on the 2nd part of the answer.]

   (c) (6 points) The height $h(t)$ (in ft. above the ground) of a passenger on a ferris wheel (a circular fair ride) varies from a maximum of 50 ft. to a minimum of 2 ft. as a function of time t (in minutes). If the ferris wheel makes 0.1 revolutions/minute, and the passenger is initially at the top of the ride, find a formula for the vertical velocity of the passenger, $v(t)$.

   The vertical velocity $v(t)$ is the derivative of the vertical position $h(t)$, so we must find that first. To find $h(t)$, we can find the midline by taking the average of the max and min, which in this case ends up being $(50 + 2)/2 = 26$. One way to find the amplitude is to subtract the midline from the maximum, so $50 - 26 = 24$. Also, since t is measured from the top of the ride (the maximum value of $h(t)$), then $h(t)$ is most easily modeled by a cosine function. Lastly, since the ride completes 0.1 revolutions/minute, then it completes 1 revolution in 10 minutes, making the period 10 minutes, from which we get the coefficient of t to be $2\pi/10 = \pi/5$. Putting all this together, we have $h(t) = 24\cos(\frac{\pi}{5}t) + 26$. with t in minutes. Thus, $v(t) = -\frac{24\pi}{5}\sin(\frac{\pi}{5}t)$.

4. (8 points) Determine a and b for the function of the form $y = f(t) = at^2 + b/t$, with a local minimum at (1,12).

Differentiating $y = at^2 + b/t$, we have

$$\frac{dy}{dt} = 2at - bt^{-2}.$$

Since there must be a critical point when $t = 1$, we have $2a - b = 0$, so $b = 2a$. Substituting and using the point (1,12) in the original equation, we have $12 = a + b = a + 2a = 3a$. Thus $a = 4$, and $b = 2(4) = 8$, and $y = 4t^2 + 8/t$.

To show that the point (1,12) is a local minimum, we use the second derivative:

$$\frac{d^2y}{dt^2} = 8 + \frac{16}{t^3},$$

which is positive for $t = 1$. Thus, the critical point (1,12) is indeed a local minimum.

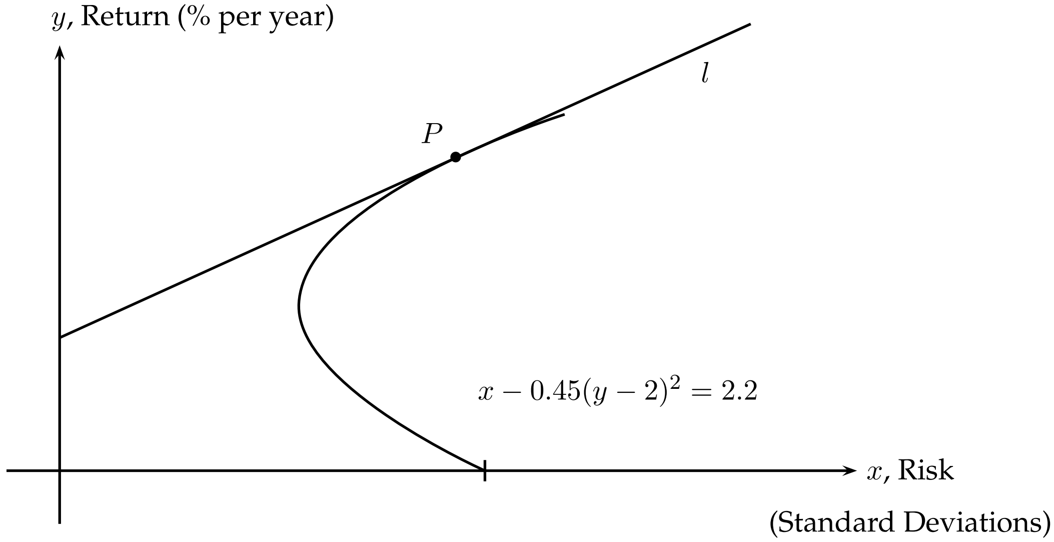
5. (6 points) The circulation time of a mammal (that is, the average time it takes for all the blood in the body to circulate once and return to the heart) is proportional to the fourth root of the body mass of the mammal. The constant of proportionality is 17.40 if circulation time is in seconds and body mass is in kilograms. The body mass of a certain growing child is 45 kg and is increasing at a rate of 0.1 kg/month. What is the rate of change of the circulation time of the child?

If we let C represent circulation time in seconds and B represent body mass in kilograms, we have $C = 17.40B^{1/4}$. As the child grows, both the body mass and the circulation time change over time. Differentiating with respect to time t, in months, we have $\frac{dC}{dt} = 17.40\left(\frac{1}{4}B^{-3/4}\frac{dB}{dt}\right)$. Substituting $B = 45$ and $dB/dt = 0.1$, we have

$$\begin{aligned}\frac{dC}{dt} &= 17.40\left(\frac{1}{4}(45)^{-3/4}(0.1)\right)\\ &= 0.025.\end{aligned}$$

The circulation time is increasing at a rate of 0.025 seconds per month.

6. In Modern Portfolio Theory, a client's portfolio is structured in a way that balances risk and return. For a certain type of portfolio, the risk, x, and return, y, are related by the equation $x - 0.45(y-2)^2 = 2.2$. This curve is shown in the graph below. The point P represents a particular portfolio of this type with a risk of 3.8 units. The tangent line, l, through point P is also shown.



(a) (5 points) Using implicit differentiation, find dy/dx, and the coordinates of the point(s) where the slope is undefined.

Taking the derivative with respect to x of the equation yields $1 - 0.9(y-2)\frac{dy}{dx} = 0$. Solving this yields

$$\frac{dy}{dx} = \frac{10}{9(y-2)},$$

which is undefined when $y = 2$. Plugging $y = 2$ into the equation of the curve, we obtain $x = 2.2$ giving $(2.2, 2)$ as the point on the curve where the slope is undefined.

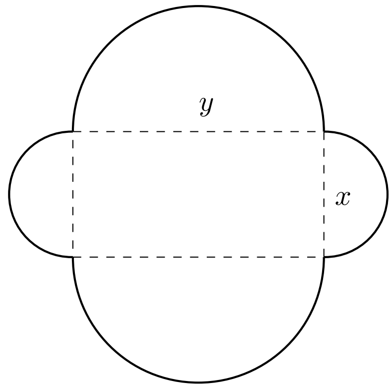
(b) (8 points) The y-intercept of the tangent line for a given portfolio is called the *Risk Free Rate of Return*. Use your answer from (a) to find the Risk Free Rate of Return for this portfolio.

We use the equation of the curve to find the y-value at P to be about 3.8856. Using this and the information from part (a) above, we get the slope of the tangent line to be about 0.5892. Thus, the equation of the tangent line can be found by using the point-slope formula, $y - 3.8856 = 0.5892(x - 3.8)$. Now, since the Risk Free Rate of Return is the y-intercept, we simply set $x = 0$ to get $y \approx 1.6464$.

(c) (3 points) Now, estimate the return of an optimal portfolio having a risk of 4 units by using your information from part (b). Would this be an overestimate or an underestimate? Why?

We can use the equation of the tangent line to approximate the return of the optimal portfolio, $y \approx 3.8856 + 0.5892(4 - 3.8) = 4.0034$. Since the graph is concave down near P, then this would be an overestimate.

7. The figure below is made of a rectangle and semi-circles.



(a) (3 points) Find a formula for the enclosed area of the figure.

The area of the figure is the rectangular area plus the two circular areas (with radius $y/2$ and $x/2$). Thus,

$$A = xy + \pi\left(\frac{x}{2}\right)^2 + \pi\left(\frac{y}{2}\right)^2.$$

(b) (2 points) Find a formula for the perimeter of the figure.

The perimeter is the sum of the four semi-circle arcs which can be joined to make two circles of diameters x and y. So the perimeter is the sum of their circumferences , or $\pi(x+y)$.

(c) (8 points) Find the values of x and y which will maximize the area if the perimeter is 100 meters.

Since this is a constrained maximization problem, we can use the constraint, $100 = \pi(x+y)$ to eliminate one variable in the area equation. Thus, we can substitute $y = \frac{100}{\pi} - x$ into A to obtain

$$A(x) = x\left(\frac{100}{\pi} - x\right) + \pi\left(\frac{x}{2}\right)^2 + \pi\left(\frac{\frac{100}{\pi} - x}{2}\right)^2.$$

Note that we know that $0 \le x, y \le \frac{100}{\pi}$.

The above equation simplifies to

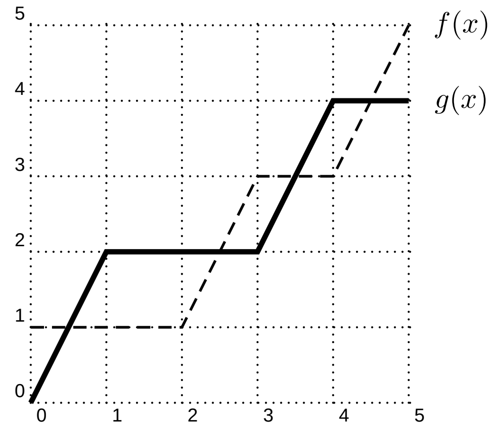
$$A(x) = \left(\frac{\pi}{2} - 1\right)x^2 + \left(\frac{100}{\pi} - 50\right)x + \frac{2500}{\pi}.$$

By solving $A'(x) = 0$, we obtain the only critical point at $x = \frac{50 - \frac{100}{\pi}}{\pi - 2}$. Along with the end-points $x = 0$ and $x = \frac{100}{\pi}$, we can evaluate the area $A(x)$ at each of these points and conclude that $A(\frac{100}{\pi})$ is the largest value. Thus, $(x, y) = (\frac{100}{\pi}, 0)$ is the location of the maximum area. [NOTE: the solution $(x, y) = (0, \frac{100}{\pi})$ works as well.]

(d) (3 points) If the cost, in dollars, of the materials to build the enclosure is given by $C(x)$ where x is in meters, and the Marginal Cost at $x = 100$ is 25, what does this mean in the context of the problem?

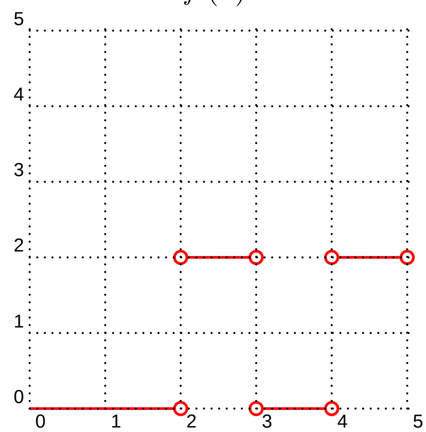
In the context of the problem, this means that the cost of the materials to build the enclosure when the perimeter is 101meters is approximately \$25 more than the cost for a 100 meter perimeter.

8. (14 points) Use the functions $f(x)$ and $g(x)$ graphed below to answer the following questions:

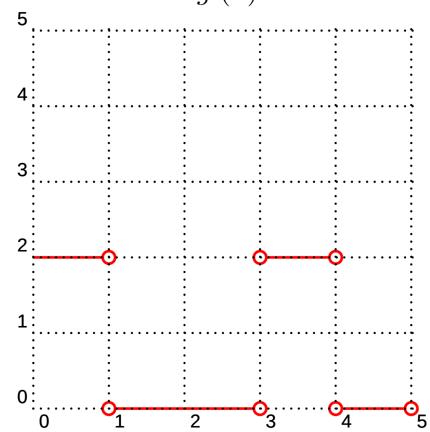


(a) (3 points each) Graph $f'(x)$ and $g'(x)$.

$f'(x)$



$g'(x)$



(b) (2 points) Compute $h'(3)$ for $h(x) = f(g(x))$.

We have that $h'(3) = f'(g(3))g'(3)$, but we see that since $g'(3)$ is undefined, that we cannot compute $h'(3)$.

(c) (2 points) Define $r(x) = g(x) - f(x)$. For what x value(s) in $[0, 5]$ is $r(x)$ maximum?

The function $r(x)$ is maximum whenever $g(x)$ is most above $f(x)$, which happens for all x in $[1, 2]$, and at $x = 4$.

(d) (2 point) Find $s'(2.5)$ for $s(x) = f(x)g(x)$.

We have that $s'(2.5) = f'(2.5)g(2.5) + f(2.5)g'(2.5)$, and using the results of part (a), we find that $s'(2.5) = (2)(2) + (2)(0) = 4$.

(e) (2 points) Find $w'(2.5)$ for $w(x) = f(x)/g(x)$.

We have that $w'(2.5) = \dfrac{f'(2.5)g(2.5) - f(2.5)g'(2.5)}{(g(2.5))^2}$, and from part (a) again, we get $w'(2.5) = \dfrac{(2)(2) - (2)(0)}{(2)^2} = 1$.