Math 115 — Final Exam

December 11, 2008

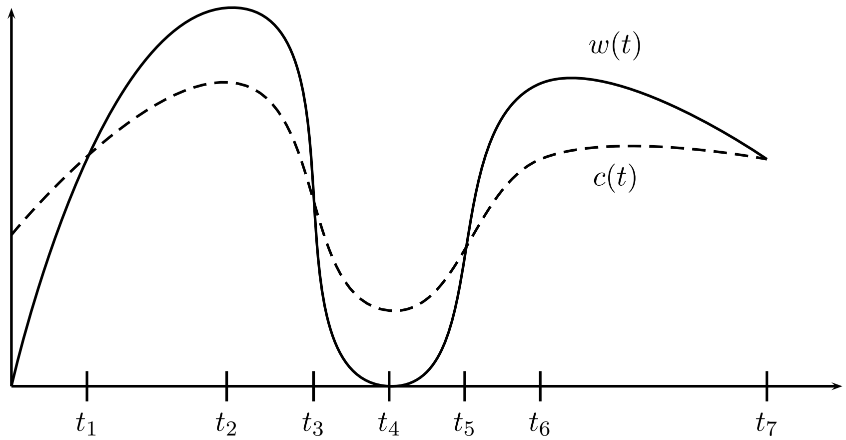
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 8 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, and it may be to your advantage to skip over and come back to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	10	
3	20	
4	11	
5	15	
6	7	
7	8	
8	12	
9	5	
Total	100	

1. [12 points] In the 17th century, a ship's navigator would estimate the distance the ship has traveled using readings of the ship's velocity, $v(t)$, in knots (nautical miles per hour). Suppose that between noon and 3:00 pm a certain galleon is traveling with the wind and against the ocean current, and that its velocity is given as the difference between the wind velocity $w(t)$ and the velocity of the ocean current $c(t)$, so that $v(t) = w(t) - c(t)$, where t is in hours since noon. Consider the wind and ocean velocities for various times between noon and 3:00 p.m., given by the graphs below:



a. [1 point] Using *integral notation* write an expression giving the distance the ship traveled from noon to 3:00 pm. Give units.

Solution: $d = \int_0^3 v(t)dt$, with the distance in nautical miles.

b. [1 point] Using *integral notation* write an expression giving the average velocity of the ship between noon and 3:00 pm. Give units.

Solution: $v_{av} = \frac{1}{3}\int_0^3 v(t)dt$, with the distance in nautical miles/hour, or knots.

c. [2 points] For what intervals was the ship's velocity positive?

Solution: The ship's velocity is positive when $w(t) > c(t)$, which happens on the intervals (t_1, t_3) and (t_5, t_7).

d. [2 points] For what t values was the ship not moving towards its destination?

Solution: Since this happens when the ship's velocity is zero or negative, the t values are $[t_0, t_1]$, $[t_3, t_5]$ and t_7.

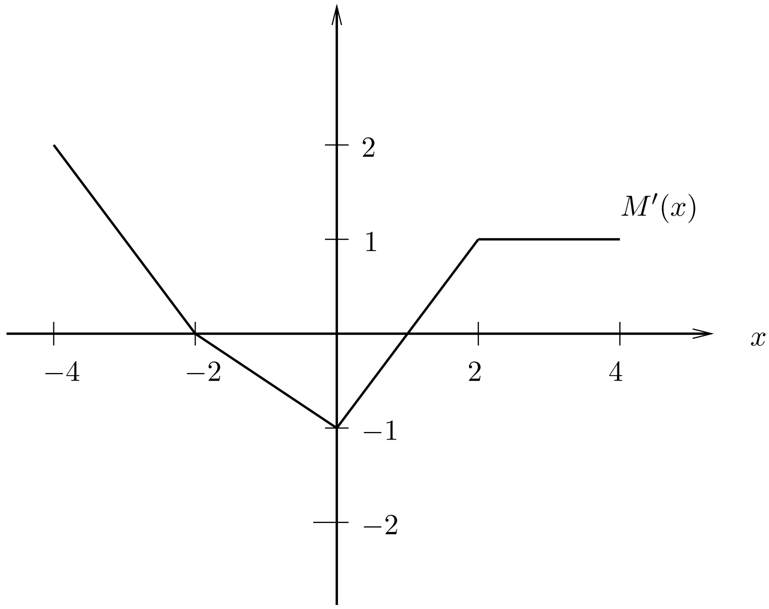
e. [2 points] For what intervals was the ship's velocity increasing?

Solution: The ship's velocity is increasing when the acceleration is positive, and since $a(t) = v'(t) = w'(t) - c'(t)$, in order for $a(t) > 0$ we need $w'(t) > c'(t)$, *i.e.* that the slope of the tangent line to $w(t)$ is greater than the slope of the tangent line to $c(t)$. This happens on the intervals (t_0, t_2) and (t_4, t_6).

f. [4 points] Please circle each integral which is positive and underline each integral which is negative.

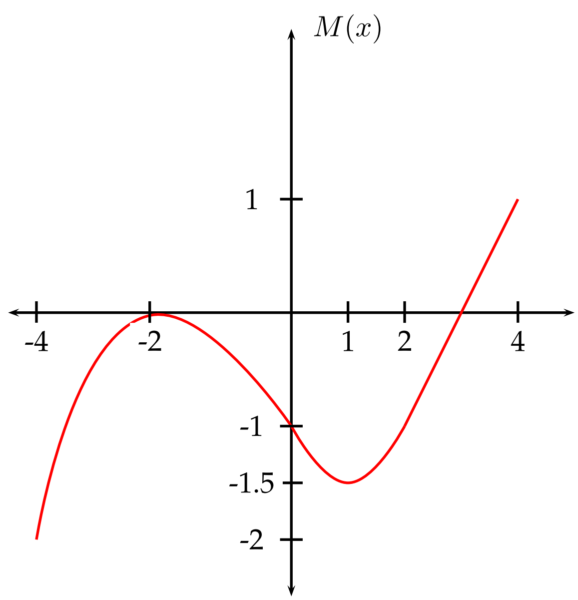
Solution: $\int_{t_1}^{t_3} v(t)dt$ $\int_{t_5}^{t_7} v(t)dt$ $\int_{t_0}^{t_7} w(t)dt$ $\int_{t_3}^{t_5} c(t)dt$

2. [10 points] The graph of the *derivative* of the continuous function $M(x)$ is given below.



Using the fact that $M(-4) = -2$, sketch the graph of $M(x)$ on the axes below. Give the coordinates of all critical points, inflection points and endpoints of M on the interval [-4,4].

Solution:



To find the y-values, we have used $M(b) = M(-4) + \int_{-4}^{b} M'(x)dx = -2 + \int_{-4}^{b} M'(x)dx$, along with the formulas of the area of a triangle and rectangle to find the integrals. For example, $M(-2) = -2 + \int_{-4}^{-2} M'(x)dx = -2 + (1/2)(2)(2) = 0$. For $-4 < x < -2$, $M'(x)$ is positive and decreasing, so $M(x)$ is increasing and concave down. We can reason similarly to determine the shape of the graph of $M(x)$ over the intervals $-2 < x < 0, 0 < x < 2$ and $2 < x < 4$, giving critical points at $(-2, 0)$ and $(1, -1.5)$. The inflection point is at $(0, -1)$, and the endpoints are $(-4, -2)$ and $(4, 1)$.

3. [20 points] The following questions are each meant to have short computation times. Each question is worth 4 points.

a. [4 points] If $f(x)$ is even and $\int_{-2}^{2}(f(-x)-3)\,dx = 8$, find $\int_{0}^{2} f(x)\,dx$.

Solution: By definition, f being even means that $f(-x) = f(x)$. Now, $\int_{-2}^{2}(f(-x) - 3)dx = \int_{-2}^{2} f(x)dx - 3\int_{-2}^{2} 1dx = \int_{-2}^{2} f(x)dx - 12$. Then, since $\int_{-2}^{2} f(x)dx = 2\int_{0}^{2} f(x)dx$, because f is even, we get that $2\int_{0}^{2} f(x)dx - 12 = 8$, which gives $\int_{0}^{2} f(x)dx = 10$.

b. [4 points] The average value of the function $g(x) = 10/x^2$ on the interval $[c, 2]$ is equal to 5. Find the value of c.

Solution: We have that $\frac{1}{2-c}\int_{c}^{2}\frac{10}{x^2}dx = 5$. Since $\int_{c}^{2}\frac{10}{x^2}dx = 10\left(\frac{-1}{2}+\frac{1}{c}\right) = 10\left(\frac{1}{c}-\frac{1}{2}\right) = 10\left(\frac{2-c}{2c}\right) = 5\left(\frac{2-c}{c}\right)$, then the left hand side of our equation becomes $\frac{1}{2-c}\left(\frac{5(2-c)}{c}\right) = \frac{5}{c}$. Thus, solving $\frac{5}{c} = 5$ yields $c = 1$.

c. [4 points] If people are buying UMAir Flight 123 tickets at a rate of $R(t)$ tickets/hour (where t is measured in hours since noon on December 15, 2008), explain in words what $\int_{3}^{27} R(t)\,dt$ means in this context.

Solution: The expression $\int_{3}^{27} R(t)dt$ represents the total number of tickets sold for UMAir Flight 123 from 3 pm on Dec. 15, 2008 to 3 pm on Dec. 16, 2008.

d. [4 points] Suppose that the function $N = f(t)$ represents the total number of students who have turned in this exam t minutes after the beginning of the exam. Interpret $(f^{-1})'(325) = 2$. [Note: this is an edit of the original problem. Grading on the final took into account the original wording.]

Solution: First note that $f^{-1}(N) = t$, so that 325 represents the number of students. Thus, $(f^{-1})'(325) = 2$ means that when 325 students have turned in their exam, it will be approximately 2 minutes before the next student turns in an exam.

e. [4 points] Find k so that the function $h(x)$ below is continuous for all x.

$$h(x) = \begin{cases} x^2+1, & x \le 1 \\ 6\sin(\pi(x-0.5)) + k, & x > 1 \end{cases}$$

Solution: Each function that makes up $h(x)$ is continuous on for all real numbers. Thus, the only place where we need to worry about $h(x)$ not being continuous is at $x = 1$. We know that $h(1) = (1)^2 + 1 = 2$, and that $\lim_{x\to 1^+} h(x) = 6\sin(\pi(1-0.5)) + k = 6 + k$. So, in order for $h(x)$ to be continuous at $x = 1$, we need that $\lim_{x\to 1^+} h(x) = h(1)$, or that $6 + k = 2$, which gives $k = -4$.

4. [11 points] Consider the function $f(x) = x^3 \ln x$.

a. [4 points] Use the general expression for a left-hand sum using 4 subdivisions to write an approximation for

$$\int_1^3 x^3 \ln x \, dx$$

—*i.e.*, express each term of the left-hand sum, using the given function. There is no need to evaluate the sum.

Solution: The width of each subdivision is given by $\Delta x = (b-a)/n = (3-1)/4 = 1/2$. We also know that the left hand sum, in this case, is given by $[f(a) + f(a+\Delta x) + f(a + 2\Delta x) + f(a+3\Delta x)]\Delta x$

$$= [1^3 \ln 1 + (1.5)^3 \ln 1.5 + 2^3 \ln 2 + 2.5^3 \ln 2.5](0.5)$$

.

b. [3 points] Show that $\displaystyle\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C$. Show your work.

Solution: The statement $\displaystyle\int f(x)dx = F(x)$ is the same as the statement $F'(x) = f(x)$, so we need to take the derivative of the right hand side $F(x) = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16} + C$ and (hopefully) get the integrand $f(x) = x^3 \ln x$. We have that

$$F'(x) = x^3 \ln x + \frac{x^4}{4}\left(\frac{1}{x}\right) - \frac{x^3}{4} = x^3 \ln x + \frac{x^3}{4} - \frac{x^3}{4} = x^3 \ln x,$$

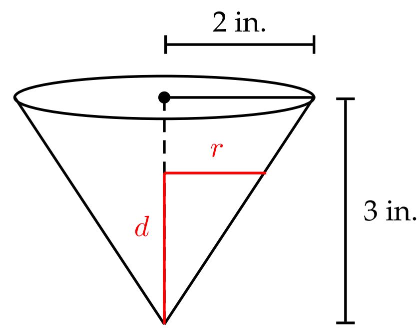
so, indeed, $\displaystyle\int x^3 \ln x \, dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$.

c. [4 points] Use the Fundamental Theorem of Calculus and part (b) to find the exact value of $\displaystyle\int_1^3 x^3 \ln x \, dx$. Leave your answer in *exact* form—in other words, do not convert to a decimal. Again, show your work.

Solution: By the Fundamental Theorem of Calculus, we know that $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$. Applying this, and part (b), we have that $\displaystyle\int_1^3 x^3 \ln x \, dx = \left(\frac{x^4}{4}\ln x - \frac{x^4}{16}\right)\Bigg|_1^3$

$$= \frac{3^4}{16}(4 \ln 3 - 1) + \frac{1}{16}.$$

5. [15 points] Suppose that you are brewing coffee and that hot water is passing through a special, cone-shaped filter (see below). Assume that the height of the conic filter is 3 in. and that the radius of the base of the cone is 2 in. If the water is flowing out of the bottom of the filter at a rate of 1.5 in^3/min when the remaining water in the filter is 2 in. deep, how fast is the depth of the water changing at that instant?
[Note: if d is depth of the water in the cone and the radius is r, the volume is given by $V = \frac{1}{3}\pi r^2 d$.]



Solution: Since the volume depends on two variables, we must eliminate one using similar triangles. As shown, we can relate r and d by $\dfrac{r}{d} = \dfrac{2}{3}$, or $r = \dfrac{2}{3}d$. Using this in V, we get

$$V(d) = \frac{1}{3}\pi\left(\frac{2}{3}d\right)^2 d = \frac{4}{27}\pi d^3.$$

From this we can find $\dfrac{dV}{dt}$:

$$\frac{dV}{dt} = \frac{dV}{dd}\cdot\frac{dd}{dt} = \frac{4}{9}\pi d^2\frac{dd}{dt}.$$

We're given that $\dfrac{dV}{dt} = -1.5$ in^3/min, and that $d = 2$. Thus, plugging in gives

$$-1.5 = \frac{4}{9}\pi(2)^2\frac{dd}{dt} = \frac{16\pi}{9}\cdot\frac{dd}{dt},$$

which allows us to solve for $\dfrac{dd}{dt}$ to get

$$\frac{dd}{dt} = -\frac{13.5}{16\pi} \approx -0.26857 \text{ in/min}.$$

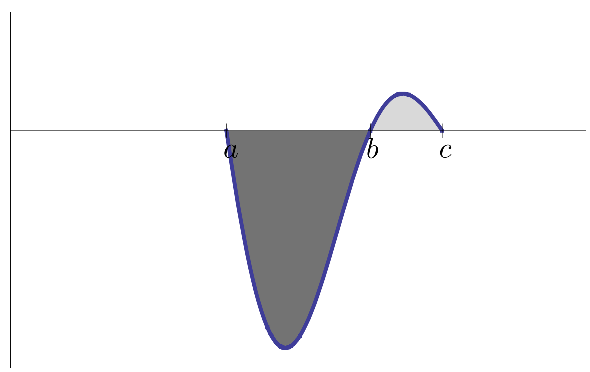
6. [7 points] The derivative of a continuous function g is given by

$$g'(x) = \frac{e^{-2x}(x+2)(x-3)^2}{(x-5)^{1/3}}.$$

Determine all critical points of g, and classify each as a local maximum, a local minimum, or neither. Carefully explain your reasoning for each classification.

Solution: Since we are already given the derivative of a continuous function, we need only find out where the derivative is zero or undefined. Since the factor e^{-2x} is positive for all x, the derivative is equal to zero at $x = -2, 3$ and undefined at $x = 5$. Applying the first derivative test around $x = -2$, we can use the equation to determined that g' changes sign from positive to negative around $x = -2$. Thus, $x = -2$ is a local maximum. Around $x = 3$, we find that $g' < 0$ both to the left and right of $x = 3$, so the critical point $x = 3$ is neither a local maximum or minimum. Lastly, around $x = 5$, the derivative changes sign from negative to positive. Thus, g has a local minimum at $x = 5$.

7. [8 points] Use the following figure, which shows a graph of $f(x)$, to find each of the indicated integrals, given that the first area (with the darker shading) is 12 units and the second area is (with the lighter shading) is 3 units.



(a) $\int_a^b f(x)\,dx$ $\underline{\quad -12 \quad}$

(b) $\int_a^c |f(x)|\,dx$ $\underline{\quad 15 \quad}$

(c) $\int_c^a f(x)\,dx$ $\underline{\quad 9 \quad}$

(d) $\int_a^a 2(f(x)+3)\,dx$ $\underline{\quad 0 \quad}$

(1)

8. [12 points] At the Michigan-Ohio State basketball game this year, the Michigan Band discovers that the amount of time it spends playing "Hail to the Victors" has a direct impact on the number of points our team scores. If the band plays for x minutes, then the Wolverines will score

$$W(x) = -.48x^2 + 7.2x + 63$$

points in the game. Assume that the band can play for a maximum of 10 minutes.

a. [5 points] How long should the band play to maximize the number of points Michigan scores? Show your work and explain.

Solution: This is a global maximum problem, and we are asked to find the global max of $W(x)$ over the interval $[0, 10]$. Let's begin by finding all the critical points. $W'(x) = -0.96x^2 + 7.2$, and setting this equal to zero yields $x = 7.5$. Now, since $W(x)$ is an inverted parabola (the coefficient of x^2 is negative), then the critical point is a local and global maximum. Thus, simply by knowing that we're dealing with an inverted parabola we are now assured that $x = 7.5$ is in fact the location of the global maximum of $W(x)$ on the interval $[0, 10]$, telling us that the band should play for 7.5 minutes in order for the Wolverines to score the maximum number of points.

b. [5 points] The band affects how many points Ohio State scores as well. When the U-M band plays for x minutes the Buckeyes score

$$B(x) = -x^2 + 8x + 84$$

points in the game. Find the number of minutes the band should play to maximize the margin of victory for Michigan (*i.e.*, the points by which Michigan wins or loses). Again, please show all work.

Solution: The margin of victory for Michigan is $M(x) = W(x) - B(x) =).52x^2 - 0.8x - 21$. The function $M(x)$ is concave up everywhere (it's a parabola opening upward), so even if it has a critical point on [0,10], that critical point must be a local minimum. Thus, the global maximum of $M(x)$ on [0,10] can occur only at one of the endpoints. Checking both endpoints, we see that $M(10)$ is greater than $M(0)$, so the global maximum occurs at $x = 10$. This tells us that the band should play for 10 minutes to maximize the Wolverine's margin of victory.

c. [2 points] What will be the score of the game for the case you found in part (b)?

Solution:

Michigan: 87 points

Ohio State: 64 points

9. [5 points] —based on your score for the Calculus Concept Inventory.