Math 115 — First Midterm

October 13, 2009

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	12	
2	12	
3	12	
4	15	
5	16	
6	10	
7	6	
8	9	
9	8	
Total	100	

1. [12 points] For each problem below, circle all of the statements that MUST be true. (The three parts (a)–(c) are independent of each other. No explanations are necessary.)

a. [5 points] Suppose f is an increasing differentiable function with domain $(-\infty, \infty)$ so that $f(1) = 1$ and $f(-1) = -1$.

- ○ f is linear.
- ○ There is a number c so that $f(c) = 0$.
- ○ $\lim_{x\to 1} f(x) = 1$
- ○ $\lim_{x\to\infty} f(x) = \infty$
- ○ $f'(1) \geq 0$

b. [3 points] Suppose $g(t)$ is the mass (in grams) of mold on a wedge of cheese in a refrigerator t days after it was abandoned. This mass grows exponentially as a function of time for two weeks, when it is finally thrown away.

- ○ The graph of g is concave up.
- ○ The continuous growth rate of g is less than the daily growth rate.
- ○ The amount of time it takes for the mass of mold on the cheese to triple is 1.5 times the amount of time it takes for it to double.

c. [4 points] If $f(x) = \dfrac{g(x)}{h(x)}$ and $h(3) = 0$ then

- ○ The graph of f has a vertical asymptote at $x = 3$.
- ○ 3 is not in the domain of f.
- ○ f is not continuous on $[-2, 2]$.
- ○ $\lim_{x\to 3} f(x)$ does not exist.

2. [12 points] Suppose that the line tangent to the graph of $f(x)$ at $x = 3$ passes through the points $(1, 2)$ and $(5, -4)$.

a. [3 points] Find $f'(3)$.

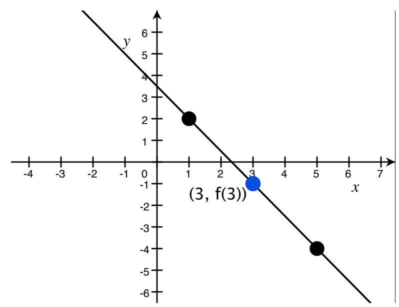
Solution:
The slope of the given tangent line is $\dfrac{\Delta y}{\Delta x} = \dfrac{-4-2}{5-1} = \dfrac{-3}{2}$.
Since $f'(3)$ IS the slope of the tangent line to the graph of $f(x)$ at $x = 3$, we find that $f'(3) = \dfrac{-3}{2}$.

b. [3 points] Find $f(3)$.

Solution:
$f(3)$ is the y–value of the point on the given line that has x–coordinate equal to 3. Using the slope found in part (a), we find that an equation for the line is

$$y = -\frac{3}{2}x + \frac{7}{2}.$$

So when $x = 3$, it follows that $y = -1$.
Hence $f(3) = -1$. (See graph to right.)

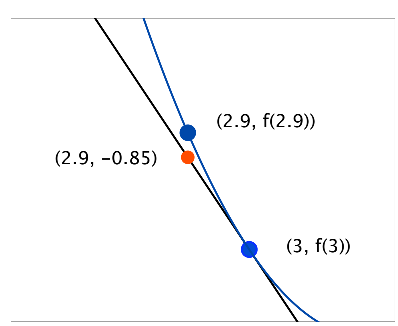


c. [3 points] Estimate the value of $f(2.9)$.

Solution: $f(2.9)$ is approximately equal to the y–value on the given line at the point $x = 2.9$. Using the equation from (b), we find $f(2.9) \approx \left(-\dfrac{3}{2}\right)(2.9) + \dfrac{7}{2} = -0.85$. Hence $f(2.9) \approx -0.85$.

d. [3 points] If the graph of f is concave up, is your estimate in part (c) an overestimate, an underestimate, or can you not tell? Explain or demonstrate your answer graphically.

Solution: Since the graph of f is concave up, its slope is increasing. So, the slope of the graph of f between $x = 2.9$ and $x = 3$ is less than (more negative) than $f'(3)$. The point on the graph of f at $x = 2.9$ will therefore be above the point on the line at $x = 2.9$. That is, $f(2.9)$ is greater than the estimate of -0.85 from part (c), so -0.85 is an UNDERESTIMATE of $f(2.9)$. (See graph to right.)

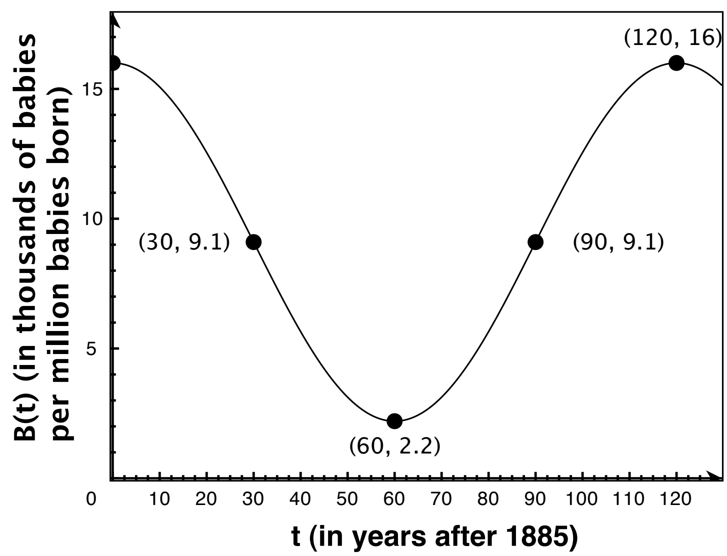


3. [12 points] The popularity of baby names varies over time; the names that are popular one year may not be popular at all within a few years. The popularity of baby names beginning with the letter I appears to be periodic. In 1885, approximately 16,000 per million babies born had first names beginning with the letter I. Their popularity began decreasing at that time and decreased until 1945, when the number had dropped to a low of 2,200 per million. In 2005 it was back to 16,000 per million babies born.

Let $B(t)$ denote the popularity of names beginning with the letter I, in thousands of babies per million babies born, t years after 1885. Assume that $B(t)$ is a sinusoidal function.

a. [6 points] Sketch the graph of $B(t)$. (Remember to clearly label your graph.)

Solution:



b. [6 points] Find a formula for $B(t)$.

Solution: The amplitude of the function is 6.9, period is 120 years (so $B = \frac{2\pi}{120} = \frac{\pi}{60}$) and the midline is at $y = B(t) = 9.1$. Since $t = 0$ corresponds to a maximum, a possible formula for $B(t)$ is

$$B(t) = 6.9\cos\left(\frac{\pi}{60}t\right) + 9.1$$

4. [15 points] The H1N1 flu virus arrived in the US last spring. Data from the CDC for the region including Michigan, Minnesota, Illinois, Indiana, Wisconsin, and Ohio is shown in the table below. $H(t)$ denotes the cumulative number of cases of H1N1 flu in this region t weeks after August 15, 2009.

t	0	1	2	3	4	5	6
$H(t)$	8266	8314	8365	8482	8632	8903	9165

a. [2 points] Evaluate and interpret $H(5)$.

Solution: $H(5) = 8903$
This means that 5 weeks after August 15, 2009 (i.e., approximately the third week of September), the cumulative number of cases of H1N1 flu in the midwest region was 8903.

b. [2 points] Why might it be reasonable to assume that H is invertible for $0 \le t \le 6$?

Solution: Since H is the *cumulative* number of cases, it is reasonable that the function is increasing with time over this interval. Thus, we could assume that H is invertible for $0 \le t \le 6$.

c. [3 points] Assuming H is invertible, give the practical meaning of $H^{-1}(8500)$.

Solution: The expression $H^{-1}(8500)$ gives us the number of weeks after August 15, 2009 in which the number of cases of H1N1 in the midwest region reached a cumulative total of 8500.

d. [3 points] Estimate $H'(5)$.

Solution: We can approximate $H'(5)$ either by taking $\dfrac{H(5) - H(4)}{5-4} = \dfrac{8903 - 8632}{5-4} = 271$, or $\dfrac{H(6) - H(5)}{6-5} = \dfrac{9165 - 8903}{6-5} = 262$, or, we could take the average of the two. Any of the answers are acceptable. The units of the answer are cases per week.

e. [5 points] Assuming H is invertible, estimate and give the practical meaning of $(H^{-1})'(8500)$.

Solution: We can approximate $(H^{-1})'(8500)$ by
$(H^{-1})'(8500) \approx \dfrac{H^{-1}(8632) - H^{-1}(8482)}{8632 - 8482} = \dfrac{4-3}{8632 - 8482} = \dfrac{1}{150} = 0.0066667$ weeks per case.
The practical interpretation here is that once we had 8500 cases in the region, the next case is likely to be reported in a little over an hour (or for the cumulative number to grow from 8500 to 8501, it would take approximately 1 hour and 7 minutes.)

5. [16 points] Since it was first introduced, the number of users of the internet worldwide has increased dramatically. Let $I(t)$ denote the number (in millions) of worldwide internet users t years after 1995. Then $I(t)$ is given by the formula

$$I(t) = \begin{cases} 16(361/16)^{t/5} & \text{if } 0 \le t \le 5 \\ 361(1.18)^{t-5} & \text{if } 5 < t \le 10 \\ A + 10(t-10) & \text{if } t > 10 \end{cases}$$

a. [3 points] Find A so that $I(t)$ is continuous.

Solution: We want to solve for A so that

$$A + 10(t-10) = 361(1.18)^{t-5} \qquad \text{when } t = 10,$$

so $A = 361(1.18)^5$.

b. [4 points] Find the continuous growth rate of $I(t)$ in the year 1997.

Solution: 1997 corresponds to $t = 2$, so the continuous growth rate is determined by the first piece of the given formula. The annual growth factor is $\frac{361}{16}^{1/5}$ (i.e. the annual growth rate is $\frac{361}{16}^{1/5} - 1$), so we need to find k so that $e^k = \dfrac{361}{16}^{1/5}$, or $k = \ln \dfrac{361}{16}^{1/5} \approx 0.62326$. Thus, the number of users was growing at a continuous rate of approximately 62% in 1997.

c. [3 points] Find the average rate of change of the number of internet users between 1995 and 2000.

Solution: Using the first definition above, we see that there were 16 million users in 1995 and 361 million users in 2000. Thus, the average rate of change for the period is $\dfrac{361 - 16}{5} = 69$ million users per year.

d. [6 points] Use the definition of the derivative to numerically estimate (i) $I'(7)$ and (ii) $I'(10)$.

Solution: (i) To approximate $I'(7)$, we use $\lim\limits_{h \to 0} \dfrac{361(1.18)^{2+h} - 361(1.18)^2}{h}$, which for small values of h from the left and the right of zero gives approximately 83.19 (to two decimal places). (ii) To approximate $I'(10)$, we must use two limits, first $\lim\limits_{h \to 0^-} \dfrac{361(1.18)^{5+h} - 361(1.18)^5}{h} \approx 136.695$, and second $\lim\limits_{h \to 0^+} \dfrac{A + 10(h) - A}{h} = 10$. Since the limit from the left of $t = 10$ is not equal to the limit from the right of $t = 10$, the limit does not exist and the function is not differentiable at $t = 10$.

6. [10 points] The force F, in Newtons, between two atoms a distance r femtometers (fm) apart in a molecule is given by $F(r) = b\left(\frac{a^2}{r^3} - \frac{a}{r^2}\right)$ for some positive constants a and b.

Note: Your answers below might involve the constants a and b.

a. [3 points] Find and interpret any horizontal intercept(s) of the graph of $F(r)$.

Solution: Horizontal intercepts occur when $F(r) = 0$. Note that $F(r) = b\left(\frac{a^2 - ar}{r^3}\right) = ba\left(\frac{a-r}{r^3}\right)$. Thus, $F(r) = 0$ when $r = a$. Under this condition, the force between the atoms is zero.

b. [3 points] Find any asymptote(s) of the graph of $F(r)$.

Solution: As $r \to \pm\infty, F(r) \to 0$, so there is a horizontal asymptote of $F(r) = 0$. Also, $F(r)$ is undefined at $r = 0$ and since the numerator is not zero there, we have a vertical asymptote at $r = 0$.

c. [4 points] Give the practical interpretation of $F'(1) = -1.2 \times 10^{-9}$.

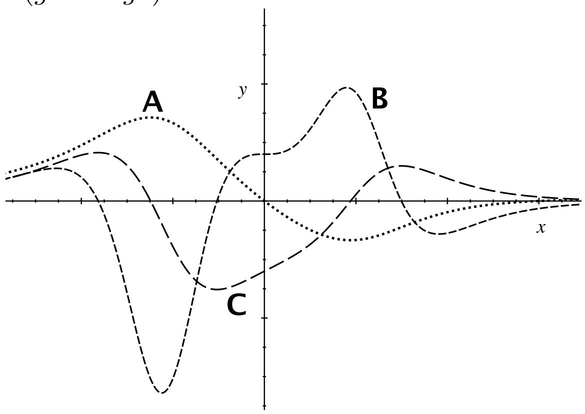
Solution: When the distance between atoms is 1 femtometer, if the distance increases another femtometer (to two femtometers), the force between the atoms would decrease by approximately 1.2×10^{-9} Newtons.

7. [6 points] Consider the function $W(t) = 3\ln\left(\sin(t)^2 + 2\right)$. Write down the limit definition of $W'(\pi)$. (You do not need to estimate or compute the derivative.)

Solution: Using the limit definition, we have

$$W'(\pi) = \lim_{h\to 0} \frac{3\ln(\sin(\pi+h)^2+2) - 3\ln(\sin(\pi^2)+2)}{h}.$$

8. [9 points] The three graphs labeled A, B, and C below depict a function g along with its first and second derivatives (g' and g''). Determine which is which.



Your answer to parts (a)–(c) should be a single legible capital letter (A, B, or C).

a. [2 points] The graph of g is labeled ______.

Solution: A

b. [2 points] The graph of g' is labeled ______.

Solution: C

c. [2 points] The graph of g'' is labeled ______.

Solution: B

d. [3 points] Briefly explain your reasoning.

Solution: Graph A cannot be the derivative of either B or C, because Graph A is positive for $x < 0$ and both Graphs B and C have intervals where the function is decreasing for $x < 0$. Thus, Graph A must be g. Graph C is positive where Graph A is increasing, negative where Graph A is decreasing and is crossing the $x-$axis at the peak and low point of Graph A. Note, also, Graph C cannot be the graph of the derivative of B, because, for example, C is negative to the left of $x = 0$ where Graph B is increasing. Graph B, however, can be the graph of the derivative of C—once again, by checking the sign of B when Graph C is increasing or decreasing, and looking for zeros of B when graph C has a peak or a valley. Thus, Graph A is g, Graph C is g', and Graph B is g''.

9. [8 points] On the axes provided below, sketch the graph of a single function f satisfying all of the following:

- $f''(x) > 0$ for $x < -2$.
- The graph of f has a vertical asymptote at $x = -2$.
- $f'(-1) = -3$
- $\lim_{x\to 0} f(x) = 2$
- $f(0) = -2$
- f is continuous but not differentiable at $x = 1$.
- $f'(x) > 0$ for $x > 3$.
- $\lim_{x\to\infty} f(x) = 4$

Remember to clearly label your graph.

Solution: A possible graph of the function is shown below: