# Math 115 — Second Midterm

November 24, 2009

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. **Use the techniques of calculus to solve the problems on this exam.**

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 8 | |
| 3 | 14 | |
| 4 | 12 | |
| 5 | 12 | |
| 6 | 14 | |
| 7 | 14 | |
| 8 | 16 | |
| Total | 100 | |

**1.** [10 points] For each of the following statements, circle **True** if the statement is *always* true and circle **False** otherwise.

**a.** [2 points] If $j'(x)$ is continuous everywhere and changes from negative to positive at $x = a$, then $j$ has a local minimum at $x = a$.

$\boxed{\text{True}}$ False

**b.** [2 points] If $f$ and $g$ are differentiable increasing functions and $g(x)$ is never equal to 0, then the function $h(x) = \dfrac{f(x)}{g(x)}$ is also a differentiable increasing function.

True $\boxed{\text{False}}$

**c.** [2 points] If $k$ is a differentiable function with exactly one critical point, then $k$ has either a global minimum or global maximum at that point.

True $\boxed{\text{False}}$

**d.** [2 points] If $F$ and $F'$ are differentiable functions and $F''(2) = 0$, then $F$ has a point of inflection at $x = 2$.

True $\boxed{\text{False}}$

**e.** [2 points] If $f$ is a differentiable function with $f(a) = b$ and $f'$ is always positive, then $f'(a)\left((f^{-1})'(b)\right) = 1$.

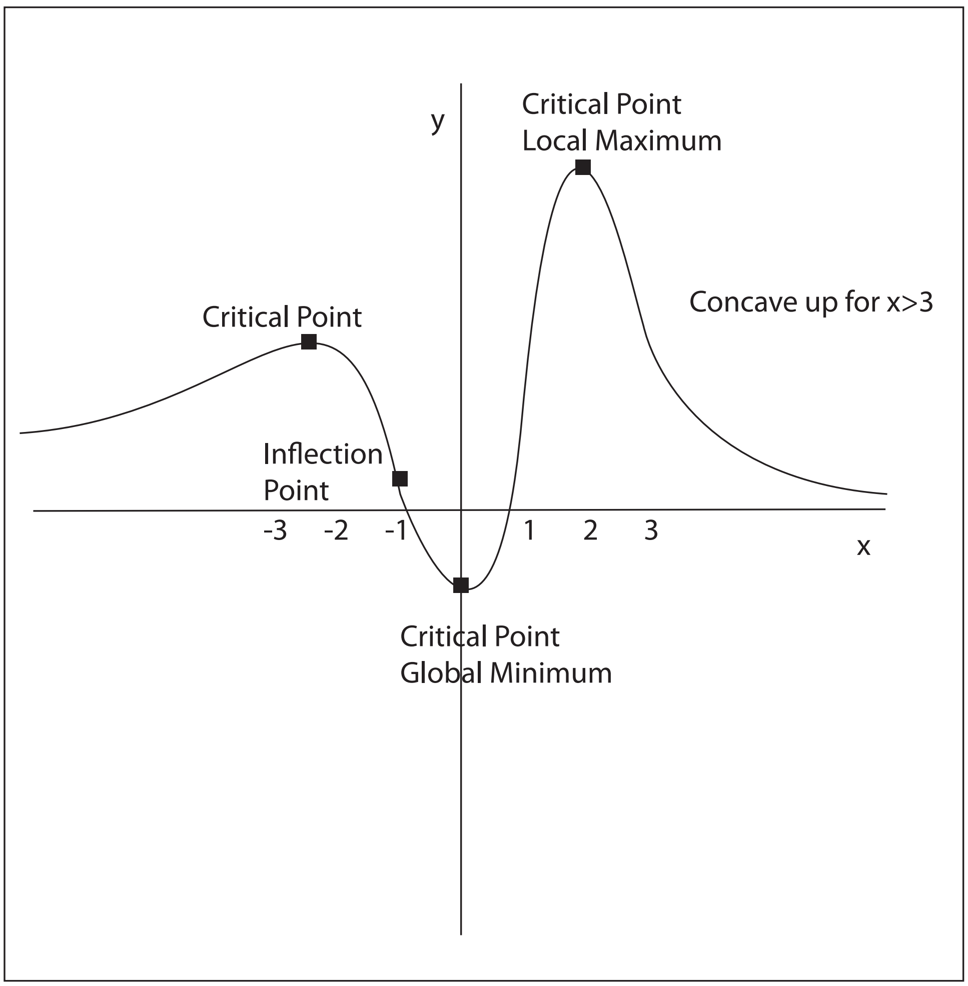
$\boxed{\text{True}}$ False

**2**. [8 points] On the axes provided below, sketch the graph of a function $f$ satisfying all of the following:

- $f$ is defined and continuous on $(-\infty, \infty)$.
- $f$ has exactly three critical points.
- $f$ has a local maximum at $x = 2$.
- $f$ has a point of inflection at $x = -1$.
- $f$ has a global minimum at $x = 0$.
- $f''(x) > 0$ for $x > 3$.

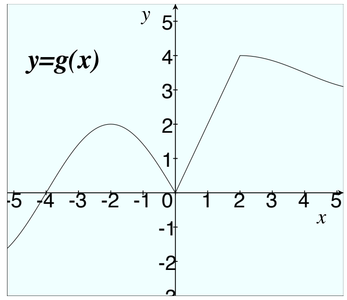
Remember to clearly label your graph. Please point out the key features.

*Solution:* Below is one possible solution.



**3**. [14 points] Use the following table and graph to answer the questions below. Note that the graph of $g$ passes through the points $(-2, 2), (0, 0)$, and $(2, 4)$. All answers should be exact.

| $x$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0 | 1 | -1 | 2 | -1 | -3 | 2 | 4 | 1 |
| $f'(x)$ | -1 | 1 | -2 | 3 | -2 | 2 | 0 | 3 | 2 |




**a**. [4 points] Let $k(x) = g(x)\arctan(f(x))$. Compute $k'(-2)$ or explain why it does not exist.

*Solution:* Using the Product and Chain Rules, we have

$$\begin{aligned} k'(x) &= g(x)\left(\frac{1}{1+(f(x))^2}\right) f'(x) + g'(x)\arctan(f(x)). \\ \text{So } k'(-2) &= g(-2)\left(\frac{1}{1+(f(-2))^2}\right) f'(-2) + g'(-2)\arctan(f(-2)) \\ &= (2)\left(\frac{1}{1+(-1)^2}\right)(-2) + 0(\arctan(-1)) = -2. \end{aligned}$$

$\boxed{\text{Hence } k'(-2) = -2.}$

**b**. [4 points] Let $a(x) = \dfrac{(f(x))^3}{3g(x)}$. Compute $a'(1)$ or explain why it does not exist.

*Solution:* Applying the Quotient and Chain Rules, we have

$$\begin{aligned} a'(x) &= \frac{3g(x)[3(f(x))^2]f'(x) - (f(x))^3(3g'(x))}{(3g(x))^2}. \\ \text{So } a'(1) &= \frac{3g(1)[3(f(1))^2]f'(1) - (f(1))^3(3g'(1))}{(3g(1))^2} \\ &= \frac{3(2)[3(-3)^2](2) - (-3)^3(3)(2)}{(3(2))^2} = \frac{27}{2} \end{aligned}$$

$\boxed{\text{Hence } a'(1) = \frac{27}{2}.}$

**c.** [6 points] Let $h(x) = g(g(x))$.
Find all local maxima and minima of the function $h$ on the interval $(-4, 4)$.
Then find the global maximum and global minimum values of $h$ on the interval $[-4, 4]$.

*Solution:* First, we find the critical points of $h$. By the Chain Rule, $h'(x) = g'(g(x))g'(x)$ so the critical points of $h$ are values of $x$ for which either $x$ or $g(x)$ is a critical point of $g$, i.e. either $g'(x)$ or $g'(g(x))$ is zero or does not exist . Critical points of $g$ are $x = -2$, $x = 0$, and $x = 2$. So, the critical points of $h$ in $(-4, 4)$ are $x = -2$, $x = 0$, $x = 2$, and $x = 1$ (since $g(1) = 2$).
We check that $h'$ is positive on the intervals $(-4, -2)$, $(0, 1)$, and $(2, 4)$ while $h'$ is negative on the intervals $(-2, 0)$ and $(1, 2)$. By the First Derivative Test, we conclude that $h$ has a local minimum at $x = 0$ and $x = 2$ and $h$ has a local maximum at $x = -2$ and $x = 1$.
Since $h$ is continuous, by the Extreme Value Theorem, $h$ attains a global maximum and a global minimum on $[-4, 4]$. These values occur at a critical point or endpoint. Evaluating $h$ at these points we have:

$$\begin{aligned}
h(-4) &= g(g(-4)) = g(0) = 0 \\
h(-2) &= g(g(-2)) = g(2) = 4 \\
h(0) &= g(g(0)) = g(0) = 0 \\
h(1) &= g(g(1)) = g(2) = 4 \\
h(2) &= g(g(2)) = g(4) \approx 3.5 \\
h(4) &= g(g(4)) \approx g(3.5) \approx 3.7
\end{aligned}$$

Thus, the global maximum occurs at $x = -2$ and $x = 1$ with a global maximum value of 4; global minimums are at $x = -4$ and $x = 0$, and the global minimum value is 0.

**4**. [12 points] In preparation for the holidays, a local bookstore is planning to sell mugs of a variety of shapes. Suppose that the amount of liquid in a "UM" mug if filled to a depth of $h$ cm is $L(h) = Uh(3M^2 - 3Mh + h^2)$ cm$^3$ for $U, M > 0$.

**a**. [4 points] Find and classify any critical points of $L$ on the interval $(0, 5M)$.

*Solution:* Taking the derivative gives

$$L'(h) = U(3M^2 - 6Mh + 3h^2) = 3U(M^2 - 2Mh + h^2) = 3U(M - h)^2.$$

Thus, the only critical point occurs at $h = M$. Note that the factor $(M - h)^2$ is positive for all $h$, so the function is increasing to the left of $h = M$ and to the right of $h = M$. Thus, the critical point is neither a local maximum nor a local minimum.

**b**. [2 points] Determine any points of inflection of $L$ on the interval $(0, 5M)$.

*Solution:* The second derivative, $L''(h) = -6U(M - h)$, shows a potential inflection point at $h = M$. The sign of the factor $-6U$ is always negative. The sign of the factor $(M - h)$ is positive to the left of $h = M$ and negative to the right. Thus, the product gives us $L''(h) < 0$ for $h < M$, and $L''(h) > 0$ for $h > M$, and the function changes from concave down to concave up at $h = M$, so $L$ has an inflection point at $h = M$.

**c**. [6 points] Suppose you are pouring coffee into a "UM" mug at a rate of 15 cm$^3$ per second. At what rate is the depth of the coffee in the mug changing when the coffee reaches a depth of 4 cm in the mug?

*Solution:* Given $dL/dt = 15$ cm$^3$/s, we want to find $dh/dt$ when $h = 4$ cm. We know

$$\frac{dL}{dt} = \frac{dL}{dh} \cdot \frac{dh}{dt},$$

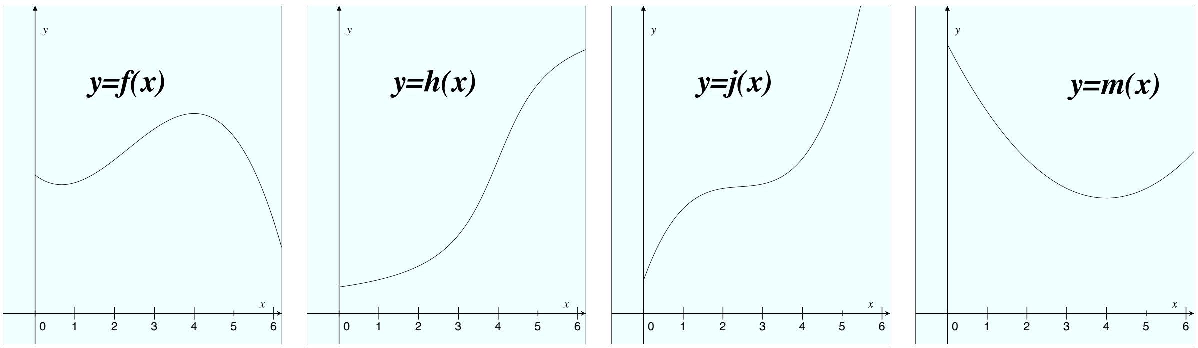
so, when $h = 4$, we have

$$15 = 3U(M - 4)^2 \cdot \frac{dh}{dt}$$

and

$$\frac{dh}{dt} = \frac{15}{3U(M - 4)^2} \text{ cm/second.}$$

**5**. [12 points] The questions (a)–(d) refer to the functions whose graphs are depicted below.




For each question, circle the **one** best answer.
Note that these questions are independent of each other; information provided in one should NOT be used in the others. *No explanations are necessary.*

**a**. [3 points] Suppose $f(x) = g'(x)$ for some differentiable function $g$. Then

i. $g$ has a local minimum at $x = 4$.

ii. $g$ has a local maximum at $x = 4$.

iii. $\boxed{g \text{ has an inflection point at } x = 4.}$

iv. None of the above

**b**. [3 points] If a function $k$ has a local maximum at $x = 2$, which of the functions above could be the second derivative of $k$?

i. $f$ ii. $h$ iii. $j$ iv. $m$ v. $\boxed{\text{None of these}}$

**c**. [3 points] Suppose the average cost (in dollars per shirt) of printing a new style of maize U of M Math Department t-shirts is smallest when 400 t-shirts are printed. One of the functions graphed above gives the total cost of printing $x$ hundred maize U of M t-shirts. Which of the graphs represents this function?

i. $f$ ii. $h$ iii. $\boxed{j}$ iv. $m$

**d**. [3 points] Suppose each of the graphs represents the derivative of a function. Which is (are) the derivative(s) of a function whose global minimum for $0 \le x \le 5$ occurs at $x = 0$?

i. $h$ and $j$ ii. $m$ iii. None of the graphs iv. $\boxed{\text{All of the graphs}}$

**6**. [14 points] The force $F$ due to gravity on a body at height $h$ above the surface of the earth is given by

$$F(h) = \frac{mgR^2}{(R+h)^2}$$

where $m$ is the mass of the body, $g$ is the acceleration due to gravity at sea level ($g < 0$), and $R$ is the radius of the earth.

**a**. [3 points] Compute $F'(h)$.

*Solution:*

$$F'(h) = \frac{-2mgR^2}{(R+h)^3}$$

**b**. [3 points] Compute $F''(h)$.

*Solution:*

$$F''(h) = \frac{6mgR^2}{(R+h)^4}$$

**c**. [5 points] Find the best linear approximation to $F$ at $h = 0$.

*Solution:* Since $F(0) = mg$ and $F'(0) = -2mgR$, the best linear approximation to $F$ at $h = 0$ is given by the equation

$$L(h) = mg - \frac{2mg}{R} \cdot h.$$

Hence, near $h = 0$, the linear approximation gives

$$F(h) \approx mg - \frac{2mg}{R} \cdot h.$$

**d**. [3 points] Does your approximation from part (c) give an overestimate or an underestimate of $F$? Why?

*Solution:* Since $F''$ is negative for all $h$ (due to $g < 0$), the function is concave down, so the tangent lies above the curve and the estimate is an overestimate.

**7.** [14 points] The *Kampyle of Eudoxus* is a family of curves that was studied by the Greek mathematician and astronomer Eudoxus of Cnidus in relation to the classical problem of doubling the cube. This family of curves is given by

$$a^2x^4 = b^4(x^2 + y^2).$$

where $a$ and $b$ are nonzero constants and $(x, y) \neq (0, 0)$—*i.e.*. the origin is not included.

**a.** [5 points] Find $\dfrac{dy}{dx}$ for the curve $a^2x^4 = b^4(x^2 + y^2)$.

*Solution:* Using implicit differentiation, we have

$$4a^2x^3 = 2b^4x + 2b^4y\frac{dy}{dx} \quad \text{so} \quad \frac{dy}{dx} = \frac{4a^2x^3 - 2b^4x}{2b^4y}.$$

**b.** [5 points] Find the coordinates of all points on the curve $a^2x^4 = b^4(x^2 + y^2)$ at which the tangent line is vertical, or show that there are no such points.

*Solution:* If the tangent is vertical, the slope is undefined. Setting the denominator from part (a) equal to zero gives $y = 0$. Substituting $y = 0$ in the original equation gives

$$a^2x^4 = b^4x^2$$

and since $(0, 0)$ is excluded, we know $x \neq 0$ so $x^2 = \dfrac{b^4}{a^2}$, and $x = \pm\dfrac{b^2}{a}$.
Thus, there are two points on the curve where the tangent is vertical: $\left(\dfrac{b^2}{a}, 0\right), \left(-\dfrac{b^2}{a}, 0\right)$.

**c.** [4 points] Show that when $a = 1$ and $b = 2$ there are no points on the curve at which the tangent line is horizontal.

*Solution:* Using $a = 1$ and $b = 2$ in $\dfrac{dy}{dx}$ from part (a), we have

$$\frac{dy}{dx} = \frac{4x^3 - 32x}{32y}.$$

If the tangent is horizontal, the slope is zero, so solving $4x^3 - 32x = 4x(x^2 - 8) = 0$ gives $x = 0$ or $x = \pm\sqrt{8}$. We must see if any of these values of $x$ give a point on the curve. Note that when $x = 0$, $y = 0$—and this point has been excluded from the family. If $x = \pm\sqrt{8}$, the equation of the curve gives us $64 = 16(8) + 16y^2$, which gives $y^2 = -4$, so there is no such point on the curve. Thus, there are no horizontal tangents to the curve $x^4 = 16(x^2 + y^2)$.

**8**. [16 points] Some airlines have started offering wireless internet access during flight. The company *WiFi Up High* (WFUH) would like to enter the market and begin working with airlines to offer such service. WFUH will charge passengers based on the amount of time the passenger uses the service during flight.
Preliminary research indicates that during NW flight 2337 (which flies from Detroit to Los Angeles), the number of hours of wifi that will be used by passengers at a price of $p$ dollars per hour is given by the function

$$h(p) = \frac{45000}{149 + e^{0.4p}}.$$

Since offering wifi to its customers is likely to increase business for the airline, WFUH also plans to charge the airline a flat fee of $a$ dollars per flight on which the service is offered.

**a**. [3 points] If WFUH offers its service to passengers at a price of \$2 per hour, what will be its expected revenue from one NW 2337 flight?

*Solution:* Since WFUH receives \$$a$ from the airline and expects to sell $h(2)$ hours at a price of \$2 per hour, its expected revenue will be $a + 2h(2) = a + 2\dfrac{45000}{149 + e^{0.8}}$, which is approximately $a + 595.14$ dollars.

**b**. [8 points] Use calculus to determine how much WFUH should charge passengers per hour of usage in order to maximize its revenue from NW flight 2337. (Round your answer to the nearest \$0.01.)

*Solution:* If WFUH charges $p$ dollars per hour, its expected revenue, in dollars, from flight 2337 is $M(p) = a + ph(p) = a + p\dfrac{45000}{149 + e^{0.4p}}$. So, we should maximize $M(p)$ on its domain, which is $[0, \infty)$. To maximize $M(p)$, we first determine its critical points.

$$M'(p) = p\frac{-45000(0.4)e^{0.4p}}{(149 + e^{0.4p})^2} + \frac{45000}{149 + e^{0.4p}} = \frac{-45000(0.4pe^{0.4p}) + 45000(149 + e^{0.4p})}{(149 + e^{0.4p})^2}.$$

Since the denominator is never 0, $M'(p)$ exists for all $p$, so the only critical points of $M$ occur when $-45000(0.4pe^{0.4p})+45000(149+e^{0.4p}) = 0$, i.e. when $149+e^{0.4p}-0.4pe^{0.4p} = 0$. Using a graphing calculator to find the horizontal ($p$–) intercept of this function, we find the critical point $p \approx 9.823$.
To see that this is the only critical point, let $f(p) = 149 + e^{0.4p} - 0.4pe^{0.4p}$. Then $f'(p) = 0.4e^{0.4p} - 0.4pe^{0.4p}(0.4) - 0.4e^{0.4p} = -0.16pe^{0.4p}$, so $f'(p) < 0$ for all $p > 0$. Hence $f$ is a decreasing function on $[0, \infty)$. It follows that the only $p$–intercept of $f$ for $p > 0$ is $p \approx 9.823$. Now $M'(9) > 0$ and $M'(10) < 0$, so $M$ has a local maximum at $p \approx 9.823$ by the First Derivative Test. Since this is the only critical point, $M$ has a global maximum at $p \approx 9.823$. $\boxed{\text{To maximize revenue, WFUH should charge \$9.82 per hour of usage.}}$
(Note that $M(9.82) > M(9.83)$, so the rounded answer is the best possible choice.)

**c**. [5 points] Suppose that WFUH initially decides to charge \$12.50 per hour of use. Note that $h(12.50)$ is approximately 151. Suppose the marginal cost to WFUH when 151 hours of wifi are being used is \$5 per hour of use. In order to increase its profit, should WFUH raise or lower the price it is charging per hour of use on NW flight 2337? Explain.

*Solution:* Marginal revenue (in dollars per hour of use) is given by $R'(h)$, where $R(h)$ is the revenue from $h$ hours of use. Note that $R(h) = a + ph$, so $R'(h) = p'(h)h + p(h)$. So when $h = 151$, we have
$R'(151) = p'(151)(151)+p(151) = \dfrac{1}{h'(12.50)}(151)+12.50 \approx 7.50$. Therefore, the marginal revenue is about \$7.50 per hour of use, so since marginal cost is only \$5 per hour of use, the profit for WFUH will increase if the number of hours of usage increases from 151 hours. Since $h'(12.5) < 0$, the number of hours of use decreases when price increases from \$12.50 per hour. So, to increase the number of hours of usage (and hence increase profits), WFUH should $\boxed{\text{decrease}}$ the price it is charging per hour of use.
Alternate Approach: Marginal cost is $\dfrac{dC}{dh}$. By the Chain Rule, $\dfrac{dC}{dp} = \dfrac{dC}{dh} \cdot \dfrac{dh}{dp}$, so when $p = \$12.50$, $\dfrac{dC}{dp} = 5 \cdot h'(12.5)$. Computing, we find that $M'(12.5) < 5 \cdot h'(12.5)$. Hence, when price increases from $p = 12.50$, profit decreases, and when price decreases from $p = 12.5$, profit increases. Therefore, WFUH should decrease the price it is charging.