Math 115 — Final Exam

December 17, 2009

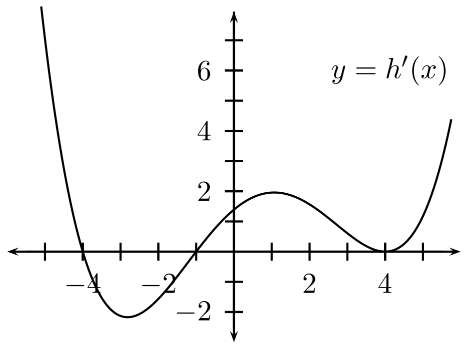
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. **Use the techniques of calculus to solve the problems on this exam.**

Problem	Points	Score
1	13	
2	14	
3	12	
4	12	
5	13	
6	12	
7	12	
8	12	
Total	100	

1. [13 points] The graph of the **derivative**, $h'(x)$, of a continuous function h is shown below:



a. [3 points] Approximate the x-coordinates of all critical points of h in the interval $(-5, 5)$, and classify each as either a local maximum, a local minimum, or neither.

Solution: Since $h'(x)$ is defined at every point in $(-5, 5)$, the critical points of h on $(-5, 5)$ are the zeros of h'. These are $x = -4$, $x = -1$, and $x = 4$. At $x = -4$, h' changes from positive to negative, so h has a *local maximum* at $x = -4$. At $x = -1$, h' changes from negative to positive, so h has a *local minimum* at $x = -1$. Since the sign of h' does not change at $x = 4$, h has *neither a local maximum nor local minimum* at $x = 4$.

b. [3 points] Approximate the x-coordinate(s) of any inflection point(s) of h in the interval $(-5, 5)$.

Solution: At (approximately) $x = -2.8$, $x = 1$, and $x = 4$, the sign of the derivative of h' changes, so the concavity of h changes. Hence h has an inflection point at each of $x = -2.8$, $x = 1$, and $x = 4$.

c. [2 points] Approximate the value(s) of x on the interval $[-5, 5]$ where h attains its global maximum.

Solution: h attains its global maximum on $[-5, 5]$ at $x = 5$.
(By comparing areas, we see that the amount by which h decreases between $x = -4$ and $x = -1$ is less than the amount by which it increases after $x = -1$.)

d. [2 points] Approximate the value(s) of x on the interval $[-5, 5]$ where h attains its global minimum.

Solution: h attains its global minimum on $[-5, 5]$ at $x = -1$.
(The area under the graph of h' between -5 and -4 is less than the area above the graph of h' between $x = -4$ and $x = -1$, so $h(-1) < h(-5)$. Then h increases after $x = -1$.)

e. [3 points] If $h(1) = 3$, find the best linear approximation to $h(x)$ at the point $x = 1$. Is this linear approximation an underestimate or an overestimate of h for points near $x = 1$? Explain.

Solution: The best linear approximation to $h(x)$ at the point $x = 1$ is given by $L(x) = h(1) + h'(1)(x - 1)$. So since $h(1) = 3$ and $h'(1) = 2$, we have $L(x) = 3 + 2(x - 1)$.
This linear approximation is an underestimate of $h(x)$ for nearby $x < 1$ (since h is concave up to the left of $x = 1$). Similarly, $L(x)$ is an overestimate of $h(x)$ for nearby $x > 1$ (since h is concave down to the right of $x = 1$).

2. [14 points] Let $C(t)$ be the temperature, in degrees Fahrenheit, of a warm can of soda t minutes after it was put in a refrigerator. Suppose $C(10) = 62$.

a. [3 points] Assuming C is invertible, give a practical interpretation of the statement $C^{-1}(45) = 40$.

Solution: It was 40 minutes after the soda was put into the refrigerator when the temperature of the soda was 45 degrees Fahrenheit. (Or the temperature of the soda was 45 degrees Fahrenheit after 40 minutes in the refrigerator.)

b. [3 points] Give a practical interpretation of the statement $C'(10) = -0.4$.

Solution: After 10 minutes in the refrigerator, the temperature of the soda would decrease by about 0.4 degrees Fahrenheit during the next minute.

c. [3 points] Give a practical interpretation of the statement $\int_0^{10} C'(t)\,dt = -5$.

Solution: The temperature of the can of soda decreased by 5 degrees Fahrenheit during the first 10 minutes it was in the refrigerator.

d. [2 points] Assuming the statements in parts (a)-(c) are true, determine $C(0)$.

Solution: By the Fundamental Theorem of Calculus, we have

$$C(0) = C(10) - \int_0^{10} C'(t)\,dt = 62 - (-5) = 67.$$

So $C(0) = 67$ degrees Fahrenheit.

e. [3 points] What is the practical meaning of $\int_0^1 C(t)\,dt$?

Solution: $\int_0^1 C(t)\,dt$ is the average temperature, in degrees Fahrenheit, of the can of soda during the first minute it is in the refrigerator.

3. [12 points]
Use the information in the table below to answer (a) - (c):

x	-3	-2	-1	0	1	2	3
$f(x)$	1	3	2	-1	-3	-2	0
$f'(x)$	3	2	-2	-3	-1	1	2
$g(x)$	2	3	1	2	4	3	1
$g'(x)$	1	-2	2	3	-1	0	-3

a. [4 points] If $h(x) = g\left(f(e^{\pi} \ln x)\right)$, find $h'(1)$. (Give an exact answer.)

Solution: By the Chain Rule, we have $h'(x) = g'(f(e^{\pi}\ln x))f'(e^{\pi}\ln x)e^{\pi}\left(\dfrac{1}{x}\right)$. So,

$$\begin{aligned} h'(1) &= g'(f(e^{\pi}\ln 1))f'(e^{\pi}\ln 1)e^{\pi}\left(\frac{1}{1}\right) = g'(f(0))f'(0)e^{\pi}(1) \\ &= g'(-1)(-3)(e^{\pi})(1) = 2(-3)(e^{\pi})(1) = -6e^{\pi}. \end{aligned}$$

b. [4 points] If $j(x) = \sin^2\left(\frac{3f(x)}{2}\right)$, find $j'(-2)$. (Give an exact answer.)

Solution: By the Chain Rule, we have $j'(x) = 2\sin\left(\dfrac{3f(x)}{2}\right)\cos\left(\dfrac{3f(x)}{2}\right)\dfrac{3}{2}f'(x)$. Thus

$$\begin{aligned} j'(-2) &= 2\sin\left(\frac{3f(-2)}{2}\right)\cos\left(\frac{3f(-2)}{2}\right)\frac{3}{2}f'(-2) \\ &= 2\sin\left(\frac{9}{2}\right)\cos\left(\frac{9}{2}\right)\frac{3}{2}(2) = 6\sin\left(\frac{9}{2}\right)\cos\left(\frac{9}{2}\right)\ (= 3\sin 9). \end{aligned}$$

c. [4 points] Give an exact answer for $\displaystyle\int_{-3}^{2} \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,dx$, assuming $g(x) \neq 0$.

Solution: By the Fundamental Theorem of Calculus and Quotient Rule,

$$\int_{-3}^{2} \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\,dx = \frac{f(2)}{g(2)} - \frac{f(-3)}{g(-3)} = \frac{-2}{3} - \frac{1}{2} = -\frac{7}{6}.$$

4. [12 points]

a. [5 points] If the average value of a continuous function g on $[1, 8]$ is 3, find

$$\int_{-1}^{6} 3\,(g(x+2)) + 5\,dx.$$

Solution: We are given that

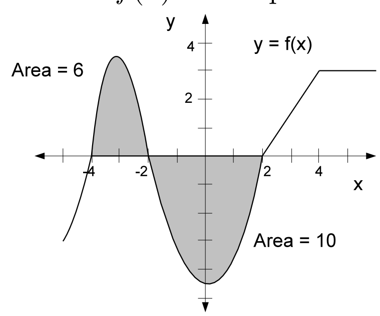
$$\frac{1}{8-1}\int_{1}^{8} g(x)\,dx = 3, \text{ so, } \int_{1}^{8} g(x)\,dx = 21.$$

$$\text{Thus, } \int_{-1}^{6} 3\,(g(x+2)) + 5\,dx = 3\int_{-1}^{6} g(x+2)\,dx + \int_{-1}^{6} 5\,dx,$$

which gives

$$3\int_{1}^{8} g(x)\,dx + 5(6-(-1)) = 3(21) + 35 = 98.$$

Use the following graph of a function $f(x)$ to compute the quantities in parts (b)–(d) below.



b. [2 points] $\int_{-4}^{2} f(x)\,dx$

Solution: $\int_{-4}^{2} f(x)\,dx = 6 - 10 = -4.$

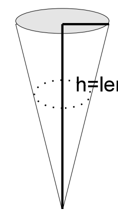
c. [3 points] The area between the graph of $f(x)$ and the x–axis for $-4 \le x \le 5$ if the units on the axes are centimeters.

Solution: Area $= 6 + 10 + 3 + 3 = 22$ cm^2.

d. [2 points] $\int_{3}^{5} f'(x)\,dx$

Solution: $\int_{3}^{5} f'(x)\,dx = f(5) - f(3) = 3 - \frac{3}{2} = \frac{3}{2}.$

5. [13 points] A cone-shaped icicle is dripping from above the entrance to Dennison Hall. The icicle is melting at a rate of 1.2 cm^3 per hour. At 10:00 a.m., the icicle was 25 cm long and had a 2 cm radius at its widest point. Assume that the icicle keeps the same proportions as it melts. [Note: the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]



a. [5 points] Determine the rate at which the length of the icicle is changing at 10:00 a.m.

Solution: We are given $\frac{dV}{dt} = -1.2$ and we want to find $\frac{dh}{dt}$ at 10:00 a.m. The proportions of the cone stay the same, so at any given time, $\frac{r}{h} = \frac{2}{25}$, so $r = \frac{2h}{25}$. Thus, $V = \frac{1}{3}\pi\left(\frac{2h}{25}\right)^2 \cdot h = \frac{1}{3}\pi\frac{4h^3}{625}$. Taking the derivative gives, $\frac{dV}{dt} = \frac{4\pi h^2}{625}\frac{dh}{dt}$, and at 10:00 a.m. we have

$$-1.2 = \frac{4\pi}{625}(25)^2\frac{dh}{dt} \text{ so that } \frac{dh}{dt} = \frac{-1.2}{4\pi} = -0.0955 \text{ cm/hour.}$$

b. [4 points] At what rate is the radius of the icicle changing at 10:00 a.m.?

Solution: From part (a), we have that $r = \frac{2h}{25}$, so $\frac{dr}{dt} = \frac{2}{25}\frac{dh}{dt}$. Thus, using $\frac{dh}{dt} = -0.0955$, we have $\frac{dr}{dt} = \frac{-0.0955(2)}{25} = -0.00764$ cm/hour.

c. [4 points] Let $V(t)$ and $r(t)$ denote the volume and radius, respectively, of the icicle t hours after 10:00 a.m. Assume that the icicle continued to melt from $t = 0$ (10:00 a.m.) to $t = M$. Circle all of the statements below that must be true if "After the icicle began dripping at 10:00 a.m., it took exactly M hours for the icicle to melt completely." [Circle the entire expression, and be certain that your circled answers are VERY clear!!]

i. $\int_0^M V'(t)\,dt > \int_0^{M/2} V'(t)\,dt$

ii. $\int_0^M V'(t)\,dt = 0$

iii. $\boxed{\int_0^M V'(t)\,dt = -V(0)}$

iv. $\int_0^2 r(t)\,dt = 0$

v. $\int_0^M r(t)\,dt = -2$

vi. $\boxed{\int_0^M r'(t)\,dt = -2}$

vii. $\int_2^0 V'(r)\,dr = M$

viii. $\int_0^M h(t)\,dt = 0$

6. [12 points] The rate $q(t)$ at which cars passed through the intersection of Main Street and Huron after a football game is presented in the table below.

t (in minutes after the game ended)	0	20	40	60	80	100	120
$q(t)$ (in cars per minute)	10	15	19	21	20	17	13

a. [4 points] What is the meaning of $\int_0^{120} q(t)\,dt$? Using a left Riemann sum and $n = 6$, estimate $\int_0^{120} q(t)\,dt$. (Write out the terms of your sum.)

Solution: The expression $\int_0^{120} q(t)\,dt$ gives the total number of cars that passed through the intersection in the first two hours after the game. A left-hand approximation with 6 subdivisions is given by

$$(20)(10 + 15 + 19 + 21 + 20 + 17) = 2040 \text{ cars} .$$

b. [2 points] Write an expression for the average rate at which cars passed through the intersection for the first two hours after the game ended.

Solution: The average rate at which cars passed through the intersection during this time period is given by $\frac{1}{120}\int_0^{120} q(t)\,dt$.

c. [3 points] Estimate $q'(30)$.

Solution: The best estimate we can get from the table is

$$q'(30) \approx \frac{19-15}{40-20} = 0.2 \text{ cars per minute per minute.}$$

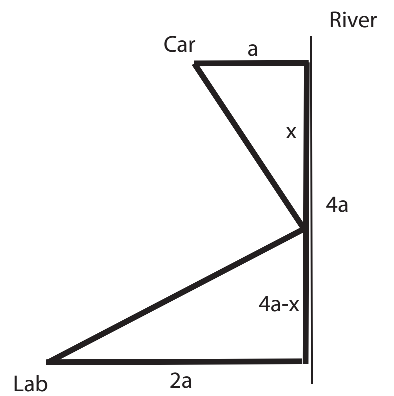
d. [3 points] If $Q(t)$ denotes the total number of cars that have passed through the intersection t minutes after the game ended, find and interpret $Q'(60)$.

Solution: We can read $Q'(60)$ from the table. We have $Q'(60) = 21$ and indicates that one hour after the game, approximately 21 additional cars would pass through the intersection in the next minute.

7. [12 points] After an unusual winter storm, the EPA is concerned about potential contamination of a river. A new researcher has been assigned the task of taking a sample to test the water quality. She tried to get as close to the river as possible in her car, but was forced to park a feet away. She also cannot get closer to the lab by car. She needs to walk to the river, retrieve a water sample, and then walk the sample to a lab located $4a$ feet down the river and $2a$ feet from the river bank.

If the researcher wants to walk as short a distance as possible, what path should she take as she walks from her car to the river and then from the river to the lab?

Solution: The shortest path will consist of a straight line from her car to the river and then from the river to the lab. Let P be the point on the river at which the researcher takes the sample of water, and let x be the number of miles downstream P is from where she parks. (See diagram below.)



The distance she walks is then given by the function $W(x) = \sqrt{a^2+x^2} + \sqrt{(2a)^2+(4a-x)^2}$. The goal is to minimize $W(x)$ on the interval $[0, 4a]$. Note that since W is continuous, the Extreme Value Theorem guarantees that W achieves a global maximum and global minimum on $[0, 4a]$.

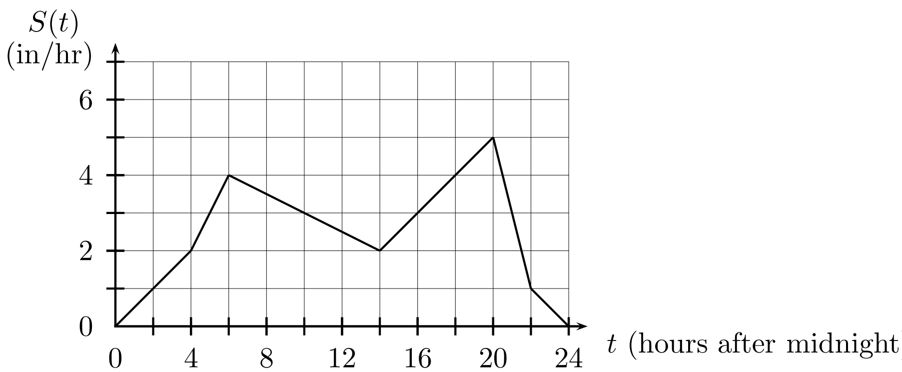
$$W'(x) = \frac{x}{\sqrt{a^2+x^2}} - \frac{4a-x}{\sqrt{(2a)^2+(4a-x)^2}}$$

Since the denominators in this expression for $W'(x)$ are never 0 (for $a > 0$), critical points in $(a, 4a)$ occur only when $W'(x) = 0$. Solving this equation, we find $\dfrac{x}{a^2+x^2} = \dfrac{4a-x}{\sqrt{(2a)^2+(4a-x)^2}}$ which reduces to $a^2(3x-4a)(x+4a) = 0$. Hence the only critical point of W on $(0, 4a)$ is $x = \dfrac{4a}{3}$.

Now, for $0 < x < \dfrac{4a}{3}$ (e.g. $x = a$), $W'(x) < 0$ and for $\dfrac{4a}{3} < x < 4a$ (e.g. $x = 2a$), $W'(x) > 0$. Hence W has a local minimum at $x = \dfrac{4a}{3}$ by the First Derivative Test. Since this is the unique critical point of W in the domain of interest, W achieves its global minimum on $[0, 4a]$ at $x = \dfrac{4a}{3}$.

So, in order to minimize the distance walked, the researcher should walk in a straight line from her car to the point on the river $\dfrac{4a}{3}$ feet downstream to retrieve a water sample and then from that point in a straight line to the lab.

8. [12 points]
The graph below gives the rate $S(t)$, in inches per hour, of snow fall t hours after midnight along a major thoroughfare in Ann Arbor. Beginning at 8:00 a.m., the city truck began removing snow at the rate of 2 in/hr. [Salting had been halted, as a consequence of economic conditions in Michigan.] Assume that there was no snow on the road prior to midnight.



a. [2 points] How deep was the snow at 2:00 p.m.?

Solution: Since the snow removal had been going on for 6 hours at 2 pm, the depth of the snow at 2:00 pm was $\int_0^{14} S(t)\,dt - 2(6) = 34 - 12 = 22$ inches. (Yes, it was a serious storm.)

b. [2 points] At what time was the snow falling the fastest?

Solution: The snow was falling fastest at $t = 20$ which is 8 pm.

c. [2 points] At what time was the snow deepest?

Solution: Since the snow removal began at 8 am, the rate at which the depth was changing is given by $S(t)$ for $0 \le t < 8$ and then by $S(t) - 2$ for $t \ge 8$. This rate is positive until $t = 21.5$ (when the line $y = 2$ intersects the graph of $S(t)$ for the second time), which is 9:30 pm. So the snow is deepest at 9:30 pm,

d. [2 points] At what time was the depth of the snow on the ground increasing fastest?

Solution: As in part (c), above, the rate at which the depth of snow is changing is given by $S(t)$ for $0 \le t < 8$ and then by $S(t) - 2$ for $t \ge 8$. We can see from the graph that this rate is greatest at time $t = 6$, i.e. at 6:00 am.

e. [2 points] What is the average rate at which snow fell between 4 am and 2 pm?

Solution: The average rate at which snow fell between 4 am and 2 pm is given by $\frac{1}{14-4}\int_4^{14} S(t)\,dt = \frac{1}{10}(30) = 3$ inches per hour.

f. [2 points] Write an expression for the average depth of the snow on the ground between 5 am and 8 am.

Solution:
Let $Q(t)$ be the depth, in inches, of snow on the ground t hours after midnight. Then the average depth of snow on the ground between 5 am and 8 am is $\frac{1}{8-5}\int_5^8 Q(t)\,dt$. Now, $Q(t) = \int_0^t Q'(x)\,dx$. As in parts (c) and (d), $Q'(t) = S(t)$ for $0 \le t < 8$. So, the average depth of snow on the ground between 5 am and 8 am is

$$\frac{1}{3}\int_5^8 Q(t)\,dt = \frac{1}{3}\left[\int_5^8 \left(\int_0^t Q'(x)\,dx\right)\,dt\right] = \frac{1}{3}\left[\int_5^8 \left(\int_0^t S(x)\,dx\right)\,dt\right].$$