Math 115 — First Midterm

October 12, 2010

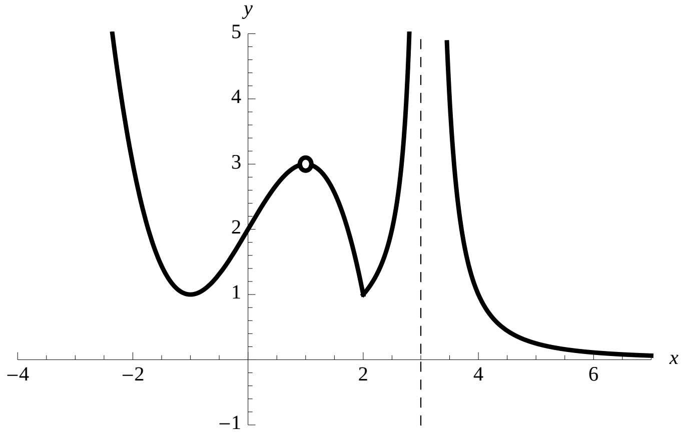
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	15	
2	12	
3	10	
4	10	
5	10	
6	11	
7	14	
8	18	
Total	100	

1. [15 points] The following figure shows the graph of $y = f(x)$ for some function f. The dotted line signifies a vertical asymptote.



a. [12 points] Using the graph, give the values of each of the following quantities if they exist. Choose your answer in each part from the numbers $0, 1, 2, 3$ or the words "Does not exist." Answers may be used more than once–or not at all.

i) $f(1) =$Does not exist.

ii) $f(2) = 1$

iii) $f(3) =$Does not exist.

iv) $f'(-1) = 0$.

v) $f'(1) =$ Does not exist.

vi) $f'(2) =$Does not exist.

vii) $\lim_{x\to+\infty} f(x) = 0$.

viii) $\lim_{x\to 3} f(x) =$Does not exist.

ix) $\lim_{x\to 2} f(x) = 1$.

x) $\lim_{x\to 1} f(x) = 3$.

xi) $\lim_{x\to -1} f(x) = 1$.

xii) $\lim_{x\to-\infty} f(x) =$ Does not exist.

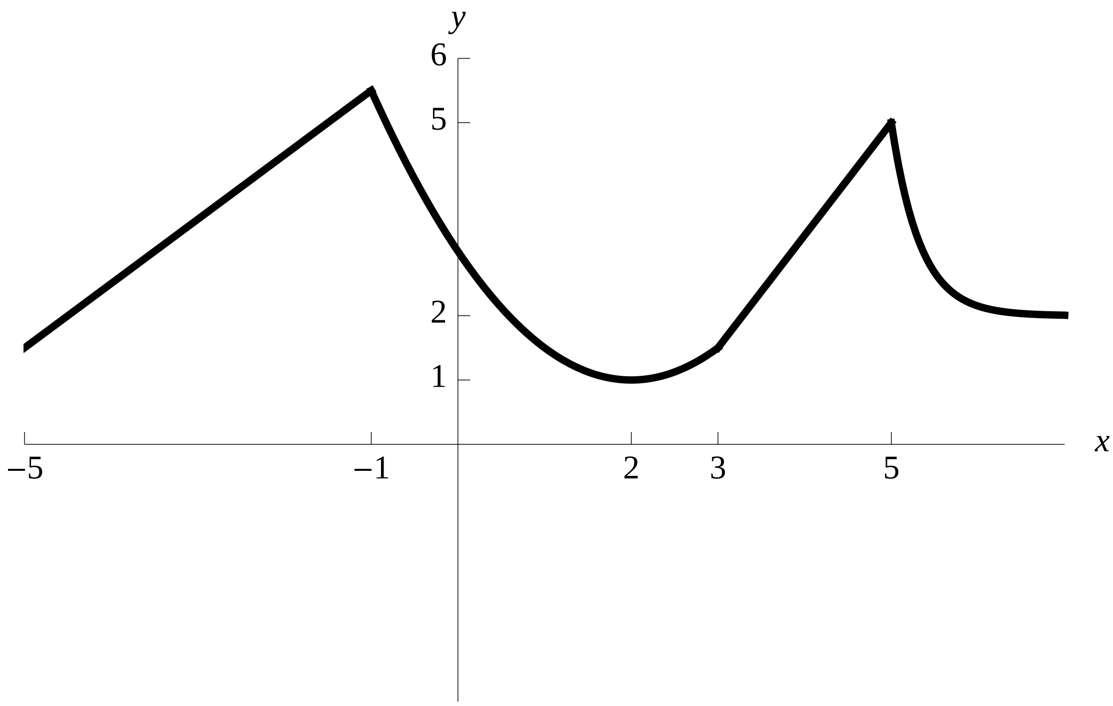
b. [3 points] Still looking at the graph, is f continuous at the following x values? (Yes or No)

i) $x = 1$, No. ii) $x = 2$, Yes. iii) $x = 3$, No.

2. [12 points] A continuous (but not necessarily differentiable) function, f, defined for all real numbers has the following properties:

- **a.** $f'(x) = 1$ for $x < -1$
- **b.** f is concave up for $-1 < x < 3$
- **c.** $f(2) = 1$
- **d.** $f'(2) = 0$
- **e.** $\lim_{x \to +\infty} f(x) = 2$
- **f.** $f''(x) > 0$ for $x > 5$

On the axes below, draw a possible sketch of $y = f(x)$ including labels where appropriate.



3. [10 points] Jim's new car came with an information sheet about the typical fuel efficiency of the car at different speeds. The fuel efficiency, E, is measured in miles per gallon (mpg) and the speed, v, is measured in miles per hour (mph). A portion of the spreadsheet is given here:

E	15	20	22.925	25	26.61	27.925
v	10	20	30	40	50	60

a. [4 points] Jim notices that, for the range of values in this table, v grows exponentially with E. Find an exponential function f so that $v = f(E)$.

Solution: Since f is exponential, it is of the form $f(E) = v_0 a^E$, where v_0 and a are constants.
Since $f(15) = 10$, we have $10 = v_0 a^{15}$. Since $f(20) = 20$, we have $20 = v_0 a^{20}$.
Solving for v_0 in terms of a in each equation we have $\frac{10}{a^{15}} = v_0 = \frac{20}{a^{20}}$.
This means $10a^{20} = 20a^{15}$.
Solving this gives $a = 2^{\frac{1}{5}}$ and $v_0 = \frac{5}{4}$.
Replacing a and v_0 with the numbers found, we have $f(E) = \frac{5}{4}(2)^{\frac{E}{5}}$.

(NOTE: This problem can also be solved with base e which gives $f(E) = 1.25e^{0.1386E}$.)

b. [3 points] Give a practical interpretation of $f^{-1}(17) = 19$.

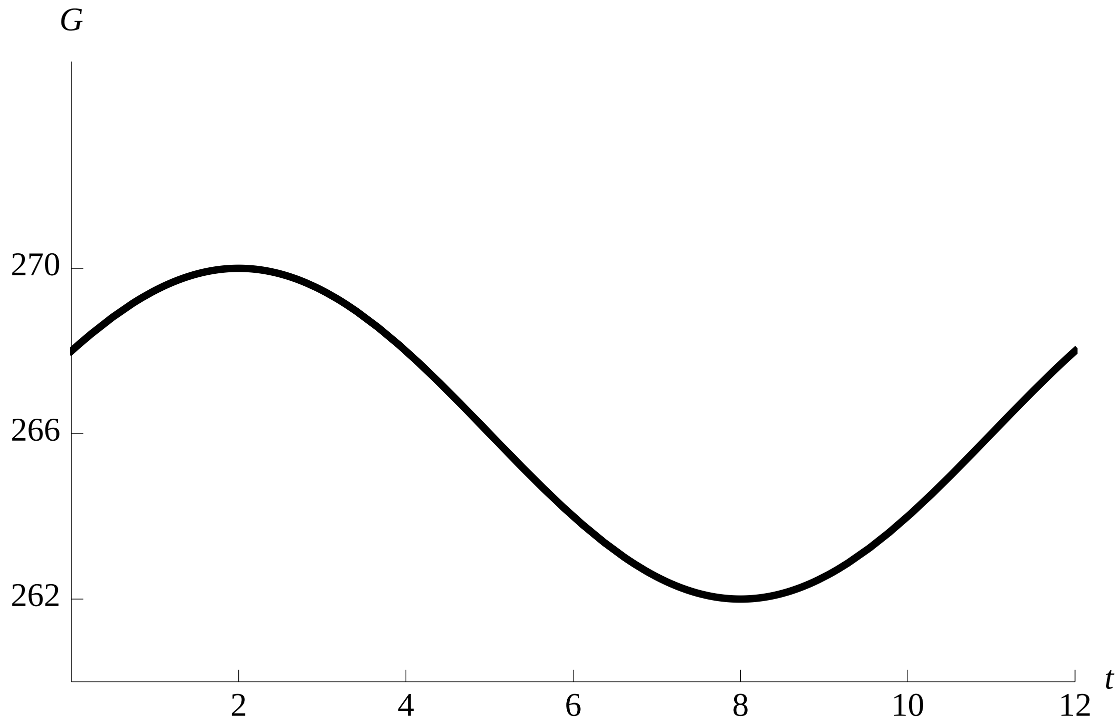
Solution: The expression $f^{-1}(17) = 19$ means "When Jim's car is going 17 miles per hour, the typical gas mileage of the car is 19 miles per gallon."

c. [3 points] Give a practical interpretation of $(f^{-1})'(25) = 0.3$.

Solution: The expression $(f^{-1})'(25) = 0.3$ means "Jim's car gets about 0.3 miles per gallon less when its speed is 25 miles per hour than when its speed is 26 miles per hour."

4. [10 points] Before the industrial era, the carbon dioxide (CO_2) level in the air in Ann Arbor was relatively stable with small seasonal fluctuations caused by plants absorbing CO_2 and producing oxygen in its place. Typically, on March 1, the CO_2 concentration reached a high of 270 parts per million (ppm), and on September 1, the concentration was at a low of 262 ppm. Let $G(t)$ be the CO_2 level t months after January 1.

a. [5 points] Assuming that $G(t)$ is periodic and sinusoidal, sketch a neat, *well-labeled* graph of G with $t = 0$ corresponding to January 1.



b. [5 points] Determine an explicit expression for G, corresponding to your sinusoidal graph above.

Solution: The function G, being a periodic, sinusiodal function, can be written in the form $G(t) = A\cos(B(t-h)) + k$. Here A is amplitude, B is $2\pi/(\textit{period})$, h is the horizontal shift, and k is the coordinate of the midline. The high point of the graph is on March 1 which corresponds to $t = 2$, so our horizontal shift will be two units to the right meaning $h = 2$. The midline is half way between the high and low values, so $k = (270 + 262)/2 = 266$. The period is 12, so $B = 2\pi/12 = \pi/6$. The amplitude is half of the difference between the high and low values, so $A = (270 - 262)/2 = 4$.
Putting all the pieces together we have

$$G(t) = 4\cos(\frac{\pi}{6}(t-2)) + 266.$$

5. [10 points] Electric cars need large amounts of energy to operate. Most types of batteries, including those found in electric cars, have reduced capacities when discharged at higher rates. For the lithium-ion batteries used in the newest electric cars, this relationship can be expressed by the equation $C = f(I) = \frac{K}{I^n}$ where C is the working capacity of the battery in amp hours (Ah) given a discharge rate of I (with $n > 1$) measured in amps (A). The constant $K > 0$ is the rated capacity of the battery.

a. [5 points] Write a formula for the derivative of C at $I = 3$ using the limit definition of the derivative. You do not need to evaluate or simplify this expression.

Solution: The limit definition of the derivative in this case is

$$\begin{aligned} \frac{dC}{dI}(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}. \\ &= \lim_{h \to 0} \frac{K/(3+h)^n - K/3^n}{h}. \end{aligned}$$

b. [3 points] Is C increasing or decreasing at $I = 3$? Justify your answer.

Solution: The capacity C is decreasing at $I = 3$. To see this, look at the expression for the slope of the secant line on C between $I = 3$ and $I = 3 + h$ for small h:

$$\frac{K/(3+h)^n - K/3^n}{h}.$$

The expression is negative if $h > 0$ because $K/(3+h)^n < K/3^n$.
The expression is also negative if $h < 0$ because $K/(3+h)^n > K/3^n$.
Since all of the secant lines near $I = 3$ have negative slope, the derivative, which is a limit of the slopes of secant lines, must also be negative.

c. [2 points] What is the concavity of the graph of C at $I = 3$? Justify your answer.

Solution: The concavity of the graph of C at $I = 3$ is concave up. As stated in the problem, the capacity C is of the form $C = \frac{K}{I^n}$. This function is a scalar multiple of the power function I^{-n} which is concave up for $I > 0$ as long as $n > 0$.

Another approach is to use the power rule to differentiate the function twice. Starting with $C = f(I) = KI^{-n}$, we have

$$f'(I) = -nKI^{-n-1}$$

and

$$f''(I) = (-n)(-n-1)KI^{-n-2} = n(n+1)KI^{-n-2}.$$

Now for $I > 0$, $f''(I) > 0$ meaning $C = f(I)$ is concave up.

6. [11 points] Aziza and Zainab are former Math 115 students at a prestigious weather forecasting company near Cloudytown, MI. Each using different meteorological instruments, they've recorded the rainfall over Cloudytown during a storm. They've let $F(t)$ be the total rainfall, in inches, t hours after the start of the storm. They collected the following data.

Aziza's data: $F(0) = 0, F(1) = 0.3$, and $F(2) = 0.5$.

Zainab's data: $F'(0) = 0.6$, $F'(1) = 0.7$, and $F'(2) = 0.3$.

a. [4 points] Use Aziza's data (and *not* Zainab's data) to estimate how quickly the rain was falling, in inches per hour, at the start of the storm (time $t = 0$) and after one hour ($t = 1$).

Solution: At the start of the storm, the rainfall rate is $F'(0)$ which can be approximated by the average rate of change of F between $t = 0$ and $t = 1$:

$$F'(0) \approx \frac{F(1) - F(0)}{1 - 0} = 0.3.$$

Similarly, the rainfall rate after one hour is approximated by

$$F'(1) \approx \frac{F(2) - F(1)}{2 - 1} = 0.2.$$

(NOTE: Several answers are acceptable for the second part.)

$t = 0$: 0.3 in/hr $t = 1$: 0.2 in/hr

b. [4 points] (True or False) Circle "T" (True) or "F" (False) for each of the statements below.

- Assuming all the data gathered was correct, throughout the second hour of the storm it was raining at a rate of about 0.7 inches per hour. F
- Assuming all the data gathered was correct, during the first hour of the storm rainfall slowed down and later sped up again. T
- Either Aziza's instrument or Zainab's instrument must be faulty because their measurements give different values for $F'(0)$ and $F'(1)$. F
- Assuming all the data gathered was correct, since $F'(0) = 0.6$ we know that about 0.6 inches of rain fell in the first hour. F

c. [3 points] Give a practical interpretation of $F'(0) = 0.6$ that begins, "During the first five minutes of the storm...".

Solution: The equation $F'(0) = 0.6$ means the rainfall rate is 0.6 in/hr at the beginning of the storm. There are twelve five minute periods in an hour, so this rate is equivalent to 0.05 inches per 5 minutes. This means our interpretation should be
"During the first five minutes of the storm, approximately 0.05 inches of rain fell."

7. [14 points] David is living on the 37th floor of a fancy building. He wants to get rid of an ancient (very energy inefficient) refrigerator that was in the building before alterations were made to the apartment. The refrigerator will not fit through the new doors of the apartment, so it must be pushed down a rather rickety ramp out the window. The ramp is 350 feet long. Below is a table showing the distance from the window along the ramp at given times:

time (seconds)	0	2	4	6	8	10	12	14	16
distance from window (feet)	0	3.9	18.6	43.1	79.4	122.5	174.6	240.1	313.6

Suppose $s(t) = d$ is the distance from the window, in feet, as a function of time, t, in seconds.

a. [3 points] Compute the average velocity of the refrigerator over the interval $4 \le t \le 12$.

Solution: The average velocity over the time interval $4 \le t \le 12$ is

$$\frac{174.6 - 18.6}{12 - 4} = 19.5 \text{ ft / sec.}$$

b. [4 points] Approximate the instantaneous velocity of the refrigerator when $t = 8$ seconds.

Solution: Avg. vel. over $6 \le t \le 8$ is $\dfrac{79.4 - 43.1}{2} = 18.15$ ft / sec..

Avg. vel. over $8 \le t \le 10$ is $\dfrac{122.5 - 79.4}{2} = 21.55$ ft / sec.

Or, we could average those velocities, giving the approximate instantaneous velocity at $t = 8$ as 19.85 ft / sec. [Note: any of the three approximations were accepted.]

c. [3 points] Approximately where will the refrigerator be after 18 seconds? Justify your answer.

Solution: The refrigerator will be on the sidewalk.

This is because the approximation of the velocity at 16 seconds is

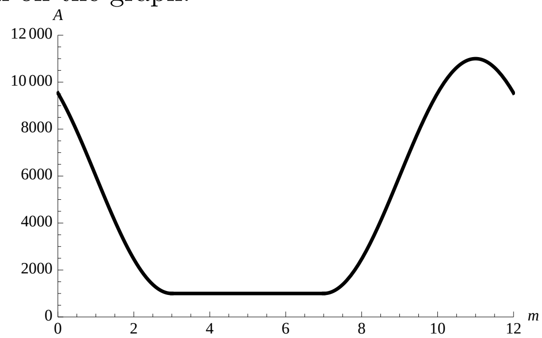
$$\frac{313.6 - 240.1}{2} = 36.75 \text{ ft / sec.}$$

which would mean that in 2 more seconds it would travel about 73.5 ft, but it is only 36.4 ft to the ground.

d. [4 points] Based upon the information in the table, does s appear to be concave up or concave down at $t = 8$? Justify your answer.

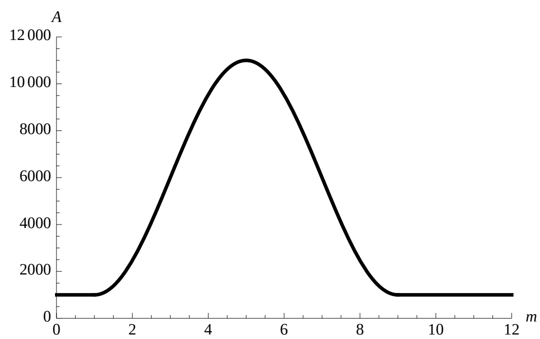
Solution: The velocity appears to be increasing at $t = 8$, because the average velocity for $6 \le t \le 8$ is less than the average velocity for $8 \le t \le 10$. Thus, if s' is increasing, s'' is positive, and the graph would be concave up.

8. [18 points] The figure below gives the graph of a function $A = b(m)$. The function is periodic and a full period is shown on the graph.



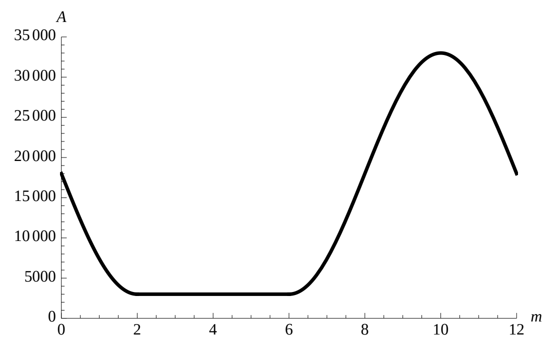
a. [8 points] For each of the following graphs, give an expression for the function depicted in terms of the function b.

$f(m)$



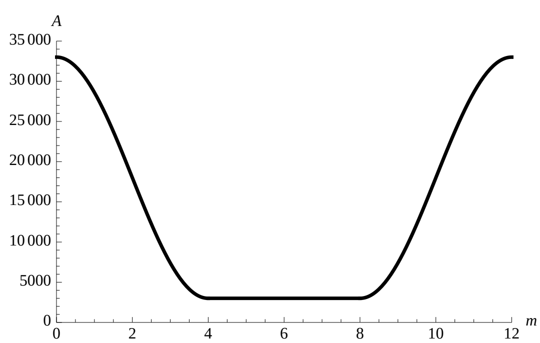
$f(m) = \underline{b(m+6)}$

$g(m)$



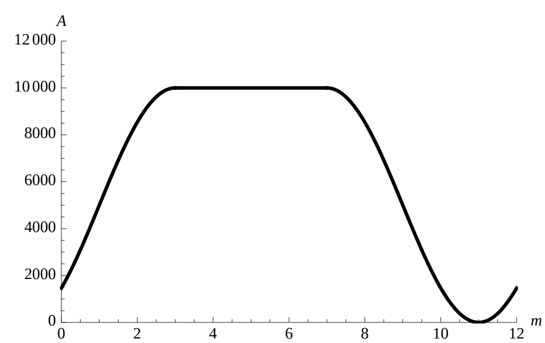
$g(m) = \underline{3b(m+1)}$

$h(m)$



$h(m) = \underline{3b(m-1)}$

$j(m)$



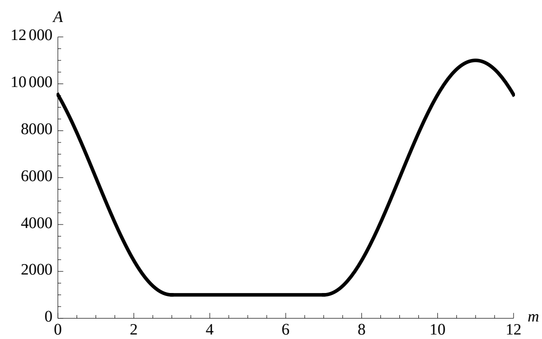
$j(m) = \underline{-b(m) + 11000}$

b. [4 points] The function b from the previous page represents the number of bushels of Michigan-grown organic apples, A, available in Michigan grocery stores as a function of the number of months, m, after January 1. The function $A = b(m)$ is repeated below.

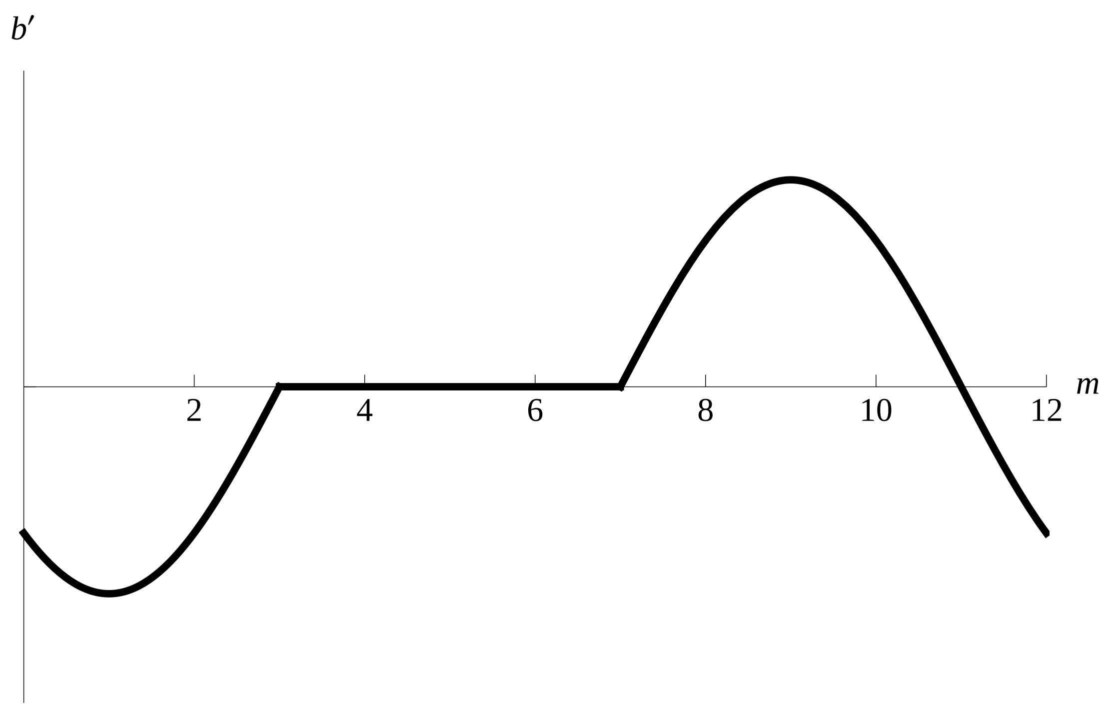
Which of the graphs on the *preceding page* could best correspond to the statement:

"In Washington, the apple growing season starts a month earlier, and the peak grocery store supply is three times as much as in Michigan." Explain your answer.

Solution: The graph of $g(m)$ best corresponds to the statement above. The graph of $g(m)$ has a peak which is three times higher than that of $b(m)$ and the graph has been shifted one unit to the left to signify the growing season beginning one month earlier.



c. [6 points] Using the graph of $b(m)$, repeated above, sketch a well-labeled graph of $b'(m)$.



Note, graphs may differ–answer is not unique.