Math 115 — Second Midterm

November 16, 2010

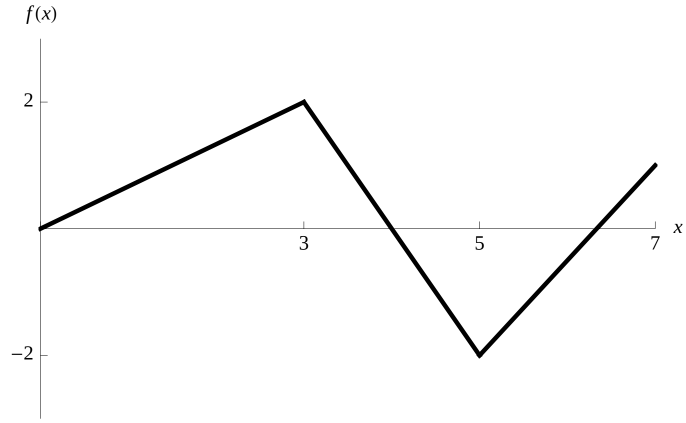
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.

Problem	Points	Score
1	10	
2	14	
3	15	
4	14	
5	12	
6	10	
7	15	
8	10	
Total	100	

1. [10 points] Given below is a graph of a function $f(x)$ and a table for a function $g(x)$.



x	0	1	2	3	4
$g(x)$	4	3	1	2	$\frac{20}{3}$
$g'(x)$	-2	$-\frac{5}{2}$	$\frac{1}{2}$	3	$-\frac{1}{3}$

Give answers for the following or write "Does not exist." No partial credit will be given.

i) $\frac{d}{dx}f(g(x))$ at $x=0$ By the chain rule $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. At $x=0$ we have

$$f'(g(0))g'(0) = f'(4)\cdot(-2) = 4.$$

ii) $\frac{d}{dx}[f(x)g(x)]$ at $x=2$ By the product rule $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. At $x=2$ we have

$$f'(2)g(2) + f(2)g'(2) = (2/3)(1) + (4/3)(1/2) = 4/3.$$

iii) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x=4$ By the quotient rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$. At $x=4$ we have

$$\frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = \frac{(-2)(20/3) - (0)(-1/3)}{(20/3)^2} = -3/10.$$

iv) $\frac{d}{dx}[g(f(x))]$ at $x=3$ We know $g'(f(3)) = 1/2$, so for values of y near $f(3)$, $g(y)$ looks like a line with slope $1/2$. So $g(f(x))$ "looks like" $\frac{1}{2}f(x)+b$ for some constant b, for x near 3. Since $f(x)$ is "pointy" at $x=3$, $\frac{1}{2}f(x)+b$ looks like a vertically compressed version of this pointy graph (near $x=3$), which is still pointy. So $g(f(x))$ is also pointy at $x=3$, hence not differentiable.

v) $f(g'(3))$ By the table, $g'(3) = 3$, so $f(g'(3)) = f(3) = 2$.

2. [14 points] The table for the *derivative* of a function h with continuous first derivative is given below. Assume that between each consecutive value of x, the derivative h' is either increasing or decreasing. For each statement below, indicate whether the statement is true, false, or cannot be determined from the information given. No partial credit will be given.

x	-4	-3	-2	-1	0	1	2	3	4
$h'(x)$	2	3	1	-3	-4	-2	0	2	1

a.) The function h has a local maximum on the interval $-2 < x < -1$.

$\boxed{\text{True}}$ False Not enough information

b.) The function h is negative on the interval $-1 < x < 1$.

True False $\boxed{\text{Not enough information}}$

c.) The function h is concave up on the interval $0 < x < 4$.

True $\boxed{\text{False}}$ Not enough information

d.) The function h is decreasing on the interval $-3 < x < -2$.

True $\boxed{\text{False}}$ Not enough information

e.) The function h has an inflection point on the interval $-1 < x < 1$.

$\boxed{\text{True}}$ False Not enough information

f.) The derivative function, h', has a critical point at $x = 2$.

True False $\boxed{\text{Not enough information}}$

g.) The second derivative function, h'', is positive on the interval $0 < x < 3$.

True False $\boxed{\text{Not enough information}}$

3. [15 points] Answer "True" or "False" for each of the following, and **include a brief explanation of your answer.** A picture may be sufficient for an explanation, if appropriate. The functions h, h', m and m' referred to in the problem are all differentiable on their domain. The letters a and b represent constants.

i) If $y = h'(x)m(x) - h(x)m'(x)$, then $\frac{dy}{dx} = h''(x)m(x) - h(x)m''(x)$.

$\boxed{\text{True}}$ False

Solution:

$$\frac{dy}{dx} = h''(x)m(x) + h'(x)m'(x) - h'(x)m'(x) - h(x)m''(x) = h''(x)m(x) - h(x)m''(x).$$

ii) If $m''(a) = 0$, then $m(x)$ has an inflection point at $x = a$.

True $\boxed{\text{False}}$

Solution: The function $f(x) = x^4$ has $f''(0) = 0$, but f'' is positive on either side of $x = 0$ so there is not an inflection point at $x = 0$.

iii) If $h''(x) > 0$ on the interval $[a, b]$ and $h(a) > h(b)$, then $h(a)$ is the absolute maximum value of $h(x)$ on $[a, b]$.

$\boxed{\text{True}}$ False

Solution: Since h is concave up for the entire interval, any critical points on the interval must be local minima. This means the maximum must occur at an endpoint (either $x = a$ or $x = b$) of the interval. The maximum value must be $h(a)$ because it is larger than $h(b)$.

iv) There exists a continuous function $f(x)$ which is not differentiable at $x = 0$ with a local maximum at $(0, 5)$.

$\boxed{\text{True}}$ False

Solution: The function $f(x) = -|x| + 5$ has a local maximum at $(0, 5)$, but has a corner at this point, so it is not differentiable at $x = 0$.

v) The function $g(x) = e^{-(x-a)^2/b}$ has a local maximum at $x = b$.

True $\boxed{\text{False}}$

Solution: The derivative of g is

$$g'(x) = -\frac{2(x-a)}{b}e^{-(x-a)^2/b}$$

so the only critical point of g is at $x = a$. Therefore there cannot be a local max at $x = b$ since it is not a critical point.

4. [14 points] It is projected that the number of marine plant and animal species on earth will decrease by 40% by the year 2050. The current (2010) instantaneous rate of marine species loss is 80,000 species per year.

a. [6 points] Assuming the number of marine species is modeled by an exponential function, write an exponential function $M = f(t)$ which outputs the total number of marine species t years after 2010.

Solution: Since the number of species M is decreasing by a constant percent, we can model with an exponential decay $M = f(t) = M_0 a^t$ with $a < 1$. If 40% of the species become extinct in 40 years, then 60% remain. Thus, $0.6M_0 = M_0 a^{40}$, and solving for a gives $a = (0.6)^{1/40}$.
The derivative is $f'(t) = M_0 \ln a \cdot a^t$. We are given $f'(0) = -80000$, so using the value of a from above, we have $M_0 = \frac{-3200000}{\ln 0.6}$. Thus, an equation for M is

$$M = \frac{-3200000}{\ln 0.6}(0.6)^{t/40}.$$

b. [3 points] According to your model, about how many marine species are there currently (in 2010)? About how many will there be in 2050?

Solution: In 2010, we have $f(0) = M_0 = \frac{-3200000}{\ln 0.6} \approx 6,264,368$ marine species in 2010.

In 2050, the model gives $f(40) = 0.6M_0 = \frac{-192000}{\ln 0.6} \approx 3,758,621$ species.

c. [5 points] In what year will there be half as many marine species as there are currently (in 2010)? Using derivatives, approximate how many marine species will be lost that year.

Solution: To find year when there will be half as many species as there are in 2010, we set $0.5M_0 = M_0(0.6)^{t/40}$. Solving this equation gives

$$t = \frac{40 \ln 0.5}{\ln 0.6} \approx 54.28.$$

There will be half as many species in 2064.
The number of species lost in that year can be approximated by

$$f'\left(\frac{40 \ln 0.5}{\ln 0.6}\right) = \frac{-3200000}{\ln 0.6}\left(\frac{1}{40}\right)(\ln 0.6)(0.6^{\frac{\ln 0.5}{\ln 0.6}}) = -40000.$$

Thus about 40,000 species will be lost in 2064.

5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At $(1, 1)$, the right hand side is $\sin(\pi) = 0$ and the left hand side is $1 - 1 = 0$. Therefore the point is on the curve since the right and left hand sides are equal.

b. [5 points] Calculate $\dfrac{dy}{dx}$.

Solution: Taking the derivative with respect to x of the equation, we have

$$\pi \cos(\pi xy) \cdot (y + x\frac{dy}{dx}) = \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{\pi y \cos(\pi xy)}{1 - \pi x \cos(\pi xy)}.$$

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution: The slope of the tangent line to the curve is

$$\frac{dy}{dx}(1, 1) = \frac{\pi \cos(\pi)}{1 - \pi \cos(\pi)} = \frac{-\pi}{1 + \pi}.$$

The equation for the tangent line is

$$y - 1 = \frac{-\pi}{1 + \pi}(x - 1).$$

6. [10 points] Calvin is stuck in the desert, and he needs to build a cube out of cactus skins to hold various supplies. He wants his cube to have a volume of 8.1 cubic feet, but he needs to figure out the side length to cut the cactus skins the right size. He has forgotten his trusty calculator, so he decides to figure out the side length of his cube using calculus.

a. [5 points] Find a local linearization of the function $f(x) = (x + 8)^{1/3}$ at $x = 0$.

Solution: The derivative is $f'(x) = \frac{1}{3}(x + 8)^{-2/3}$. To find the local linearization we compute $f'(0) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$ and $f(0) = 2$. The equation for the tangent line to f at $x = 0$ is $y - 2 = \frac{1}{12}x$. So the local linearization of f near $x = 0$ is

$$L(x) = \frac{1}{12}x + 2.$$

b. [3 points] Use your linearization to approximate $(8.1)^{1/3}$.

Solution: We need to approximate $(8.1)^{1/3} = f(0.1)$. According to our local linearization,

$$f(0.1) \approx L(0.1) = \frac{1}{12}(0.1) + 2 = \frac{241}{120}.$$

c. [2 points] Should your approximation from part **b.** be an over-estimate or an underestimate? Why?

Solution:
The second derivative of f is $f''(x) = -\frac{2}{9}(x + 8)^{-5/3}$. For values of x near 0, the second derivative will be negative which means f is concave down near 0. This means our estimate is an overestimate.

7. [15 points] Suppose a is a positive constant and

$$f(x) = 2x^3 - 3ax^2.$$

a. [10 points] Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[-a, \frac{3}{2}a]$. Specify all x values where the maximum and minimum value are achieved.

Solution: Seeking critical points, we take the derivative of f and set it equal to zero.

$$f'(x) = 6x^2 - 6ax = 6x(x-a) = 0.$$

Using this equation we find the critical points to be $x = 0, a$. Now we put the critical points and the endpoints of the interval back into the orginal function and compare the values. We compute $f(-a) = -5a^3,\ f(0) = 0,\ f(a) = -a^3,\ f(\frac{3}{2}a) = 0.$
This means the absolute max of f on this interval is 0 and this value is achieved at $x = 0, \frac{3}{2}a$. The absolute min is $-5a^3$ and this value is achieved at $x = -a$.

b. [5 points] Find all inflection points of $f(x)$.

Solution: Seeking inflection points, we find $f''(x) = 12x - 6a$. Setting this equal to zero we find $x = \frac{a}{2}$. To verify this is an inflection point we test $f''(0) = -6a < 0$ and $f''(a) = 6a > 0$. This means f'' changes sign at $x = \frac{a}{2}$, so it is an inflection point.

8. [10 points] Farmer Fred is designing a fence next to his barn for his grass-fed herd of cattle. The fence will be rectangular in shape with wooden fence on three sides and a chain link fence on the side closest to his barn. The wooden fence costs \$6 per foot and the chain link fence costs \$3 per foot. If he wants the fenced area to be 40,000 square feet, what should the dimensions of his fence be in order to minimize his total cost?

Solution: Let x be the length in feet of the fence along the the side closest to the barn (there is one side of this length made of wood and another made of chain link). Let y be the length in feet of two other sides, both of which are wood.
The area of the fence is $40000 = xy$. The cost of the fence is $C = 3x+6x+6y+6y = 9x+12y$. We can use the area equation to solve for x in terms of y.

$$x = 40000/y.$$

Then

$$C = 360000/y + 12y.$$

Now we calculate

$$\frac{dC}{dy} = -360000y^{-2} + 12.$$

Setting this expression equal to zero, we see that $y = \sqrt{30000}$ which makes $x = 40000/\sqrt{30000}$. To see that this is a minimum, we calculate

$$\frac{d^2C}{dy^2} = 720000y^{-3}$$

which is positive at the critical point we found above, showing that this critical point is a minimum of C. This minimum must be global since it's the only critical point on the domain of C. Therefore the dimensions which minimize the cost of the fence are $y = \sqrt{30000}$ ft by $x = 40000/\sqrt{30000}$ ft.