Math 115 — Final Exam

December 17, 2010

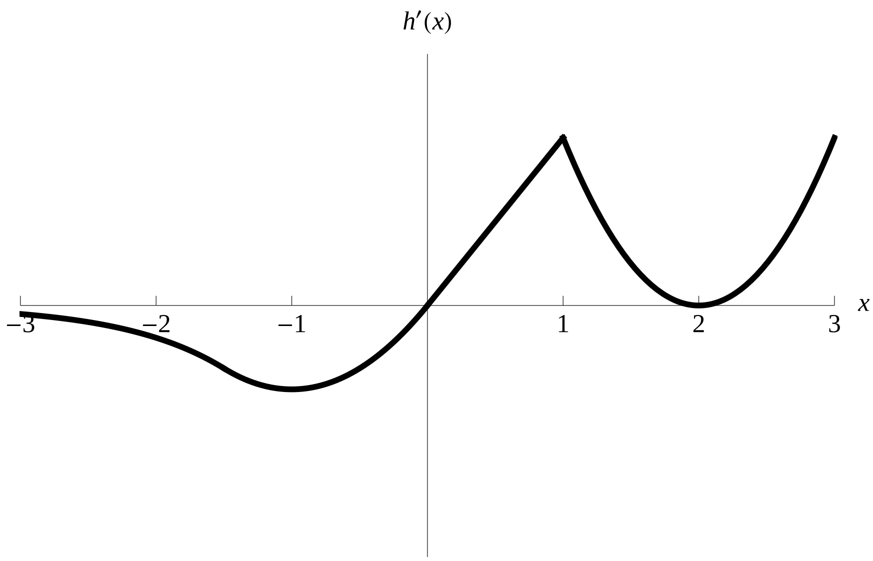
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

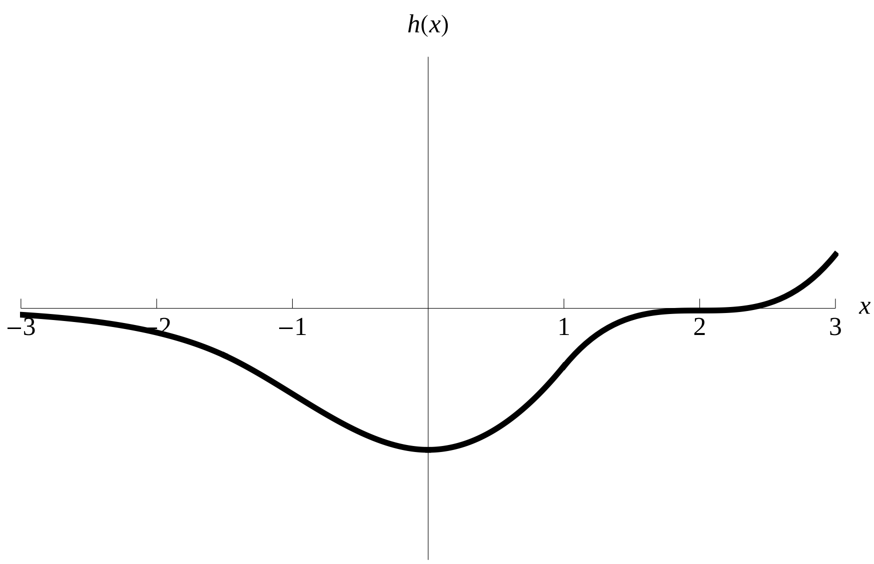
1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. **Use the techniques of calculus to solve the problems on this exam.**

Problem	Points	Score
1	10	
2	13	
3	12	
4	14	
5	12	
6	10	
7	14	
8	15	
Total	100	

1. [10 points] Given below is a graph of $h'(x)$, the derivative of a function $h(x)$.



(a) On the axes below, sketch a possible graph of $h(x)$.



(b) List the x-coordinates of all inflection points of h. $x = -1, 1, 2.$

(c) Give the x-coordinate of the global minimum of h on [-3,3]. $x = 0.$

(d) Give the x-coordinate of the global maximum of h on [-3,3]. $x = 3.$

2. [13 points] The U-value of a wall of a building is a positive number related to the rate of energy transfer through the wall. Walls with a lower U-value keep more heat in during the winter than ones with a higher U-value. Consider a wall which consists of two materials, material A with U-value a and material B with U-value b. The U-value of the wall w is given by

$$w = \frac{ab}{b+a}.$$

Considering a as a constant, we can think of w as a function of b, $w = u(b)$.

a. [4 points] Write the limit definition of the derivative of $u(b)$.

Solution: The derivative of $u(b)$ is defined to be

$$u'(b) = \lim_{h\to 0} \frac{u(b+h) - u(b)}{h} = \lim_{h\to 0} \frac{a(b+h)/(b+h+a) - ab/(b+a)}{h}$$

b. [4 points] Calculate $u'(b)$. (You do **not** need to use the limit definition of the derivative for your calculation.)

Solution: By the quotient rule,

$$u'(b) = \frac{(b+a)(a) - ab}{(b+a)^2} = \frac{a^2}{(b+a)^2}.$$

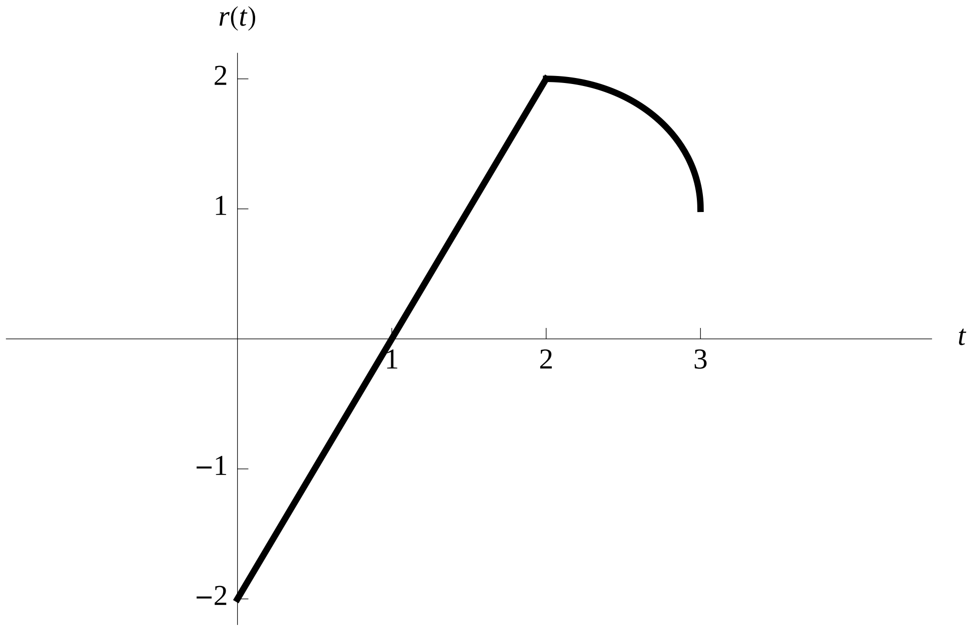
c. [5 points] Find the x- and y-coordinates of the global minimum and maximum of $u(b)$ for b in the interval $[1, 2]$. Your answer may involve the parameter a. [Recall that $a, b > 0$.]

Solution: The derivative $u'(b)$ is strictly positive for all $b > 0$ by part **b.** This means there are not any critical points on $[1, 2]$ and u is strictly increasing so $b = 1$ is the global minimum while $b = 2$ is the global maximum. Now we compute $u(1) = \frac{a}{1+a}$ and $u(2) = \frac{2a}{2+a}$.

Global minimum on $[1, 2]$: $(1, \frac{a}{1+a})$

Global maximum on $[1, 2]$: $(2, \frac{a}{2+a})$

3. [12 points] Shown below is a graph of a function $r(t)$. The graph consists of a straight line between $t = 0$ and $t = 2$ and a quarter circle between $t = 2$ and $t = 3$.



Calculate the following using the graph and the properties of integrals.

a. [4 points] $-3\int_0^3 (2 + r(t))dt$.

Solution: We compute

$$-3\int_0^3 (2 + r(t))dt = -6\int_0^3 1dt \;-\; 3\int_0^3 r(t)dt = -18 - 3(1 + \pi/4) = -21 - 3\pi/4.$$

b. [4 points] $\int_{1/2}^{3/2} r'(t)dt$.

Solution: By the fundamental theorem of calculus,

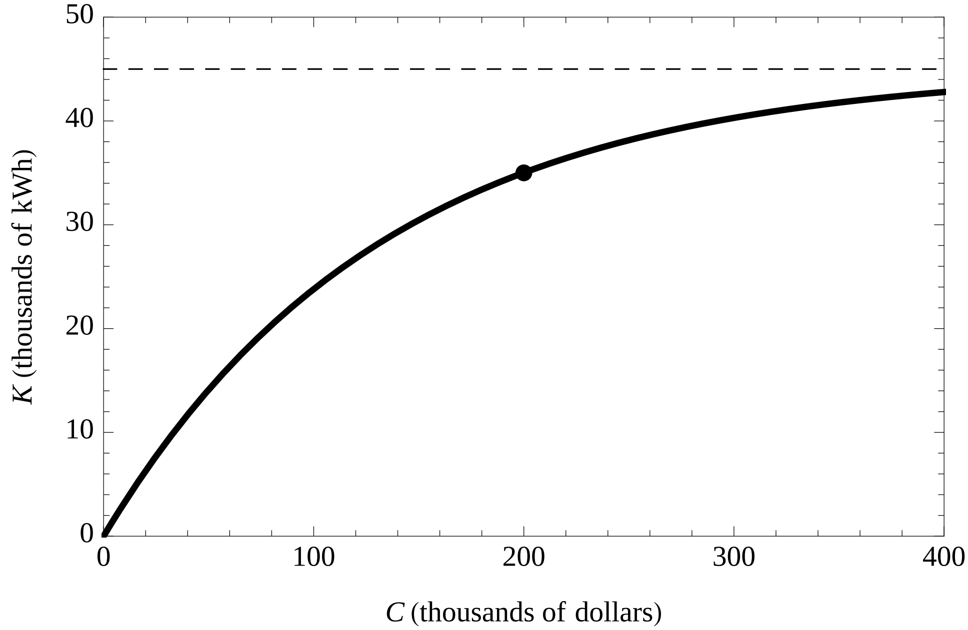
$$\int_{1/2}^{3/2} r'(t)dt = r(3/2) - r(1/2) = 1 - (-1) = 2.$$

c. [4 points] The average value of r on the interval $[1, 3]$.

Solution: The average value of r on the interval $[1, 3]$ is

$$\frac{1}{3-1}\int_1^3 r(t)dt = \frac{1}{2}(1 + 1 + \pi/4) = 1 + \pi/8.$$

4. [14 points] Business owner Abbey Alexander is constructing a building for her latest business. Abbey wants her building to be energy efficient in order to save money on utility costs. Abbey has been given the following graph to help her decide on how much to spend on improvements.



In the graph, K is the expected savings in thousands of kilowatt hours (kWh) per year if Abbey spends C thousand dollars on energy-efficiency improvements. The dark point on the curve is (200, 35) and the dotted line is a horizontal asymptote at $K = 45$.

a. [5 points] Write a function of the form $K = a(1 - e^{-bC})$ for the curve in the graph above.

Solution: Since there is a horizontal asymptote at $K = 45$, we have

$$45 = \lim_{C\to\infty} K = a.$$

To find b we set

$$35 = 45(1 - e^{-200b}).$$

Solving for b, we have $b = \frac{\ln(9/2)}{200}$. Thus our equation for K is

$$K = 45(1 - e^{-\frac{\ln(9/2)}{200}C}).$$

b. [3 points] The current price of energy from Abbey's power company is \$250 per thousand kWh. Assuming this price stays constant, write a function $F(C)$ which gives Abbey's total savings on utility costs (in thousands of dollars) over the first 20 years.

Solution: Abbey's utility savings over the first 20 years will be

$$250 \cdot 20K = 250 \cdot 20 \cdot 45(1 - e^{-\frac{\ln(9/2)}{200}C}) = 225000(1 - e^{-\frac{\ln(9/2)}{200}C}).$$

The units here are dollars. If we want to write $F(C)$ in thousands of dollars (as required below) we must divide by 1000 giving $F(C) = 225(1 - e^{-\frac{\ln(9/2)}{200}C})$.

c. [6 points] If Abbey spends C thousand dollars on energy-efficiency improvements, her net monetary savings, N, over 20 years, is given by the formula

$$N = F(C) - C$$

where $F(C)$ is from part **b.** How much should Abbey spend on energy-efficiency improvements in order to maximize her net monetary savings over the first 20 years? Be sure to justify your answer.

Solution: We take the derivative of N seeking critical points.

$$N'(C) = 1.692e^{-\frac{\ln(9/2)}{200}C} - 1.$$

Solving for C, we have $C = 69.931$. Because

$$N''(C) = (-\frac{\ln(9/2)}{200})(1.692)(e^{-\frac{\ln(9/2)}{200}C}) = (-)(+)(+) < 0$$

for all $C \geq 0$, we know that N is concave down everywhere which means our solution, $C = 69.931$, is a global maximum. Thus Abbey should spend \$69,931 on improvements.

5. [12 points] A certain type of spherical melon has weight proportional to its volume as it grows. When the melon weighs 0.2 pounds, it has a volume of 36 cm^3 and its weight is increasing at a rate of 0.1 pounds per day. [Note: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

a. [3 points] Find $\dfrac{dV}{dt}$ when the melon weighs 0.2 pounds (t measured in days).

Solution: When the melon weighs 0.2 pounds, it has a volume of 36 cm^3, so if $V = kw$ where V is volume, w is weight and k is a proportionality constant, then $36 = 0.2k$ giving $k = 180$. Now $\frac{dV}{dt} = k\frac{dw}{dt} = 180(0.1) = 18$ cm^3 per day.

b. [5 points] Find the rate at which the radius of the melon is increasing when it weighs 0.2 pounds.

Solution: We are looking for $\frac{dr}{dt}$ when $w = 0.2$. Differentiating the equation for a sphere gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We know $36 = \frac{4}{3}\pi r^3$, so $r = \frac{3}{\pi^{1/3}}$. We also know that $\frac{dV}{dt} = 18$. Putting this together we have

$$18 = 4\pi \frac{9}{\pi^{2/3}} \frac{dr}{dt}.$$

which means $\frac{dr}{dt} = \frac{1}{2\pi^{1/3}}$ cm per day.

c. [4 points] Use a local linearization to approximate the volume of the melon 36 hours after it weighs 0.2 pounds.

Solution: Since $\frac{dV}{dt} = 18$ cm^3 per day and 36 hours is 1.5 days we know the melon will increase by approximately 18(1.5)=27 cm^3 in the 36 hours after it weighs 0.2 pounds. Since it is 36 cm^3 at the time in question, the volume will be about 63 cm^3 36 hours later.

6. [10 points] The Green Bag Company (GBC) makes hand bags out of recycled materials. A table of the company's marginal cost, MC, and marginal revenue, MR, at various production levels q is given below. The variable q is the number of hand bags produced, and marginal cost and marginal revenue are measured in dollars per bag.

q	1000	2000	3000	4000	5000	6000
MC	100	81	75	96	112	123
MR	125	123	114	110	107	106

Assume for this problem that GBC's cost and revenue functions are twice differentiable and that MC and MR are either increasing or decreasing on each interval shown in the table.

a. [3 points] At which production level from the table is GBC's profit increasing the fastest? Explain your answer.

Solution: The derivative of the profit is $MR - MC$. We are looking for the place where this derivative is largest, so by inspecting the table, we see that $MR - MC$ is largest (with a value of 42) when $q = 2000$ bags.

b. [3 points] The CEO of the company thinks profit is maximized at 3000 bags, but the CFO of the company thinks that profit will be maximized at 4500 bags. Who could be correct, and why? [Note: The terms "CEO" and "CFO" refer to officers in the company.]

Solution: The CFO could be correct because the profit will be maximized when $MC = MR$ and this *could* occur at $q = 4500$ since $MR - MC$ changes sign somewhere in the interval $4000 < q < 5000$. It definitively does not occur at $q = 3000$ because $MR > MC$ at this point which means producing more bags will result in higher profits for the company.

c. [4 points] Assuming GBC has no fixed costs, use a right-hand sum to estimate the cost to produce the first 3000 bags. Be sure to show your work.

Solution: Doing a right hand sum, we have

$$C(3000) = \int_0^{3000} MC(q)dq \approx 1000MC(1000)+1000MC(2000)+1000MC(3000) = 256000.$$

So the approximation predicts the cost to produce the first 3000 bags will be \$256,000.

7. [14 points] The rate at which a coal plant releases CO_2 into the atmosphere t days after 12:00 am on Jan 1, 2010 is given by the function $E(t)$ measured in tons per day. Suppose $\int_0^{31} E(t)dt = 223$.

a. [4 points] Give a practical interpretation of $\int_{31}^{59} E(t)dt$.

Solution: $\int_{31}^{59} E(t)dt$ is the amount of CO_2 the plant releases into the atmosphere in February.

b. [4 points] Give a practical interpretation of $E(15) = 7.1$.

Solution: Since $t = 15$ corresponds to 12am on January 16th, the statement $E(15) = 7.1$ can be interpreted as
"On January 16th (or 15th) the plant releases approximately 7.1 tons of CO_2 into the atmosphere."

c. [2 points] The plant is upgrading to "clean coal" technology which will cause its July 2010 CO_2 emissions to be one fourth of its January 2010 CO_2 emissions. How much CO_2 will the coal plant release into the atmosphere in July?

Solution: Given in the problem is $\int_0^{31} E(t)dt = 223$ which means the plant released 223 tons of CO_2 into the atmosphere in January. This means that the plant will release 223/4 tons in July.

d. [4 points] Using a left-hand sum with four subdivisions, write an expression which approximates $\int_{31}^{59} E(t)dt$.

Solution: The length of the interval is 28, so with 4 subdivisions $\Delta t = 7$. This means our left hand sum is

$$\int_{31}^{59} E(t)dt \approx 7(E(31) + E(38) + E(45) + E(52)).$$

8. [15 points] In each part of this problem, write "True" on the blank line for all statements which *must* be true based on the information given. If the statement is not necessarily true, write "False."

a. [5 points] The function $g(x)$ is differentiable on $(-\infty, \infty)$ and $g'(3) = 0$.

- True The function $g(x)$ is continuous for all real values of x.
- False The function $g(x)$ has a local maximum or a local minimum at $x = 3$.
- False The second derivative of g exists at $x = 3$.
- False The derivative of $(x \cdot g(x))^2$ at $x = 3$ is equal to 0.
- True The derivative of $g(x)$ at $x = 2$ exists.

b. [5 points] A differentiable function $v(t)$ gives the velocity of a particle at a time $t \geq 0$. The function v is positive for all t in its domain.

- True The integral $\int_a^b v(t)dt$ is the total distance traveled by the particle between $t = a$ and $t = b$ for $0 \leq a \leq b$.
- True The function $v'(t)$ gives the acceleration of the particle at a time $t \geq 0$.
- False The function $v'(t)$ is positive for some value of t.
- True The average velocity of the particle between $t = 1$ and $t = 2$ is $\int_1^2 v(t)dt$.
- True The particle is traveling in the same direction at all times.

c. [5 points] Let $g(R)$ be the amount of natural gas in liters used by an R rated furnace in an hour of operation. The rating of a furnace is a number between 0 and 100 which is related to the efficiency of the furnace. The higher the rating of a furnace, the more efficient it is. Suppose $g'(95) = -0.01$, $(g^{-1})'(2) = -40$, $g(95) = 1$, and $g^{-1}(2) = 40$.

- False It is reasonable to expect that a furnace which uses one liter of natural gas in an hour has a rating which is approximately 40 more than a furnace which uses two liters in an hour.
- True It is reasonable to expect that a furnace which uses 1.9 liters of natural gas in an hour has a rating which is approximately 4 more than a furnace which uses two liters in an hour.
- True It is reasonable to expect that in one hour of operation, a furnace with a rating of 90 uses about 0.05 more liters of natural gas than a furnace with a rating of 95.
- False For each one point rating drop from a rating of 95, a furnace will use 0.01 more liters of natural gas in one hour of operation.
- True A furnace with a rating of 40 uses two liters of natural gas in an hour of operation.