Math 115 — Final Exam

December 15, 2011

Name: ____

Instructor: ______________________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	6	
3	16	
4	10	
5	8	
6	10	
7	14	
8	12	
9	10	
Total	100	

1. [14 points] You are online playing the Facebook-based game, FarmVille, and you receive land with 5 stalks of corn on it. You decide that you would like to model the corn population on this patch of land using your calculus skills, so you recall that a good model for population growth is the logistic model

$$P(t) = \frac{L}{1 + Ae^{-kt}} \qquad L > 0, \quad A > 0, \quad k > 0.$$

a. [5 points] Using the ***limit definition of the derivative***, write an explicit expression for the derivative of the function $P(t)$ at $t = 1$. Do not evaluate this expression.

b. [5 points] Using the definition of the logistic model above, compute the following in terms of L, k, and A, showing your work or providing an explanation for each part:

i. [1 points] $\lim_{t\to\infty} P(t)$

ii. [1 points] $\lim_{t\to-\infty} P(t)$

iii. [1 points] $P(0)$

iv. [2 points] $P'(0)$

c. [4 points] Your farmland satisfies the following conditions:

$$P(0) = 5, \; P'(0) = 1, \; \lim_{t\to\infty} P(t) = 100.$$

Based on your answers in part (b), compute the correct values of L, k, and A for the logistic equation modeling corn population on your land.

$L =$ ________________ $A =$ ________________ $k =$ ________________

2. [6 points] For each of the following statements, circle TRUE if the statement is always true and circle FALSE otherwise. Any ambiguous answers will be marked as incorrect.

a. [2 points] Suppose $A(t)$ and $B(t)$ are both everywhere differentiable functions which satisfy the equation $A^2 = e^B$ for all real numbers t. If, additionally, $\ln(2A(0)) = B(0)$, then $A'(0) = B'(0)$.

True False

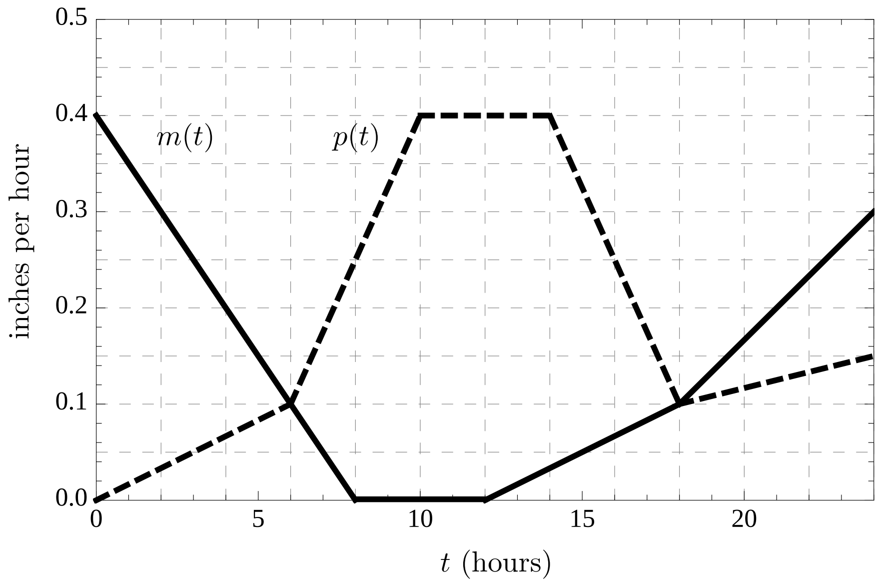
b. [2 points] If $f(x)$ is an everywhere continuous function and $\int_1^b f(t-b)dt = c$ for some real numbers b and c, then $\int_0^{1-b} 5f(t)dt = -5c$.

True False

c. [2 points] If $r(y)$ is a twice differentiable function whose first derivative is continuous, decreasing, and negative for all real numbers y, then $r(y)$ is concave up.

True False

3. [16 points] Suppose the graph below shows the rate of snow melt and snowfall on Mount Arvon, the highest peak in Michigan (at a towering 1970 ft), during a day (24 hour period) in April of last year. The function $m(t)$ (the solid curve) is the rate of snow melt, in inches per hour, t hours after the beginning of the day. The function $p(t)$ (the dashed curve) is the snowfall rate in inches per hour t hours after beginning of the day. There were 18 inches of snow on the ground at the beginning of the day.



a. [2 points] Over what time period(s) was the snowfall rate greater than the snow melt rate?

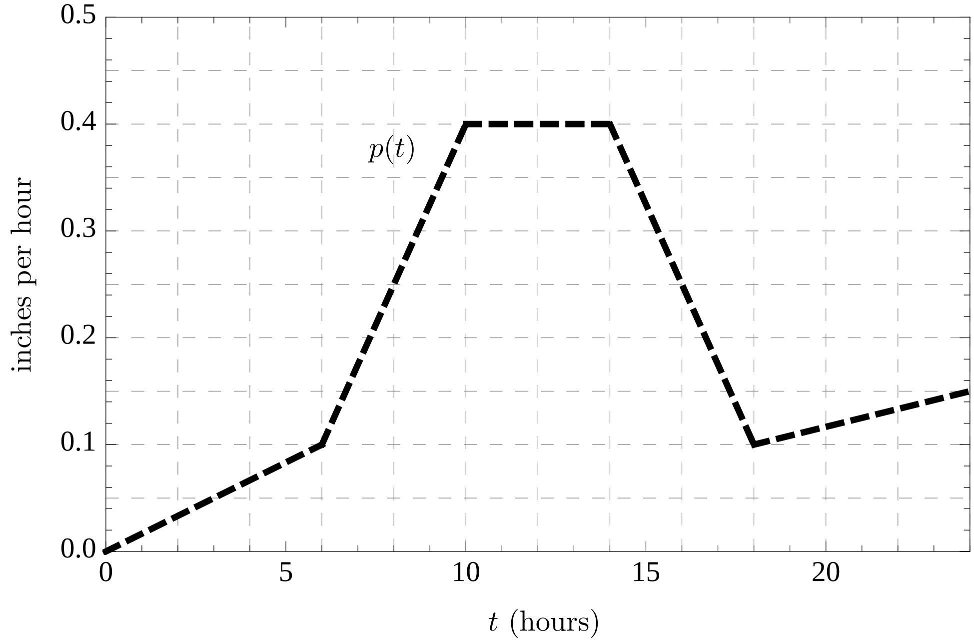
b. [2 points] When was the amount of snow on Mount Arvon increasing the fastest? When was it decreasing the fastest?

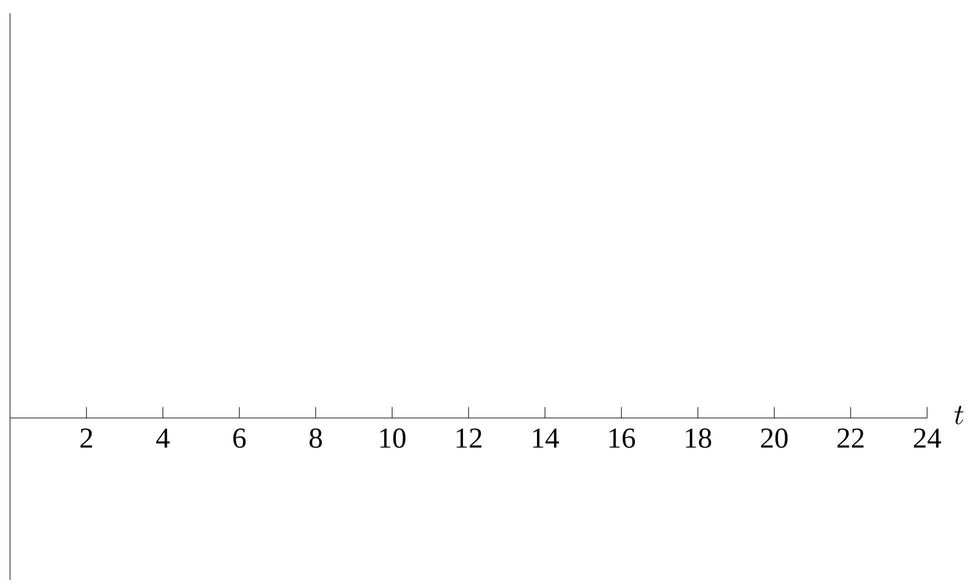
c. [3 points] When was the amount of snow on Mount Arvon the greatest? Explain.

d. [3 points] How much snow was there on Mount Arvon at the end of the day (at $t = 24$)? Show work.

3. (continued)

e. [6 points] The graph of $p(t)$ is repeated below. On the empty set of axes, sketch a well-labeled graph of $P(t)$, an antiderivative of $p(t)$ satisfying $P(0) = 0$. Label and give the coordinates of the points on the graph of $P(t)$ at $t = 10$ and $t = 18$.





4. [10 points] Your family farm has a small herd of dairy cows. You decide to track the daily milk production levels of your favorite cow, Bessie, over the course of several months. Below is your table of measurements from every two months of Bessie's milk production level $p(t)$ in liters per day:

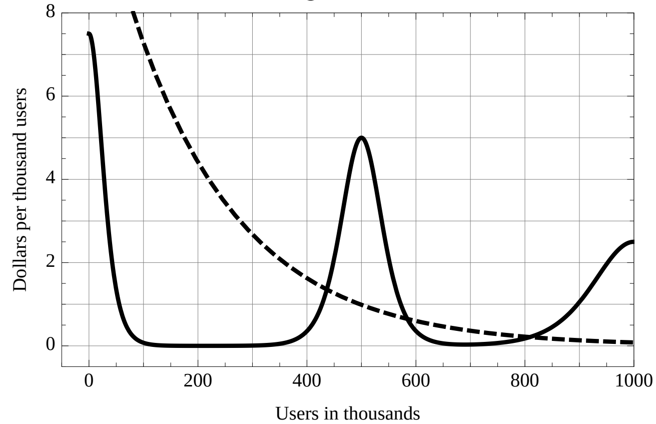
day t	0	60	120	180	240	300	360
liters per day $p(t)$	18	24	18	6	0	6	18

a. [4 points] The function modeling Bessie's daily milk production $p(t)$ is sinusoidal. What is an equation for this function?

b. [3 points] Suppose you use a right-hand Riemann sum to approximate the total milk produced by Bessie between day 60 and day 240. Is this an overestimate, an underestimate, or can it not be determined? Explain your reasoning.

c. [3 points] How many measurements would you need to take between day 60 and day 240 to be sure your right-hand Riemann sum approximation is no more than 10 liters off from the exact amount of milk Bessie produces?

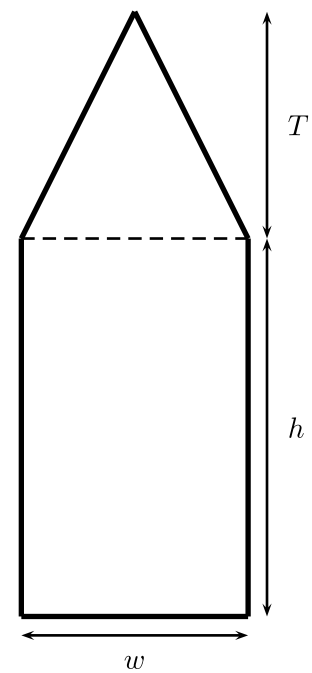
5. [8 points] The owners of a social networking site are concerned about their profit margins, so they develop the graphs for marginal revenue and marginal cost in terms of the number of users on the site. Use the graphs below to answer the following questions. The dashed graph is marginal revenue and the solid line is marginal cost.



a. [3 points] At which approximate number(s) of users is marginal cost equal to marginal revenue?

b. [5 points] Which of your answers from part (a) maximizes profit? Clearly justify your answer.

6. [10 points] Consider a window the shape of which is a rectangle of height h surmounted by a triangle having a height T that is two times the width w of the rectangle (see the figure below which is not drawn to scale). If the total area of the window is 5 square feet, determine the dimensions of the window which minimize the perimeter.



7. [14 points] For positive A and B, the force between two atoms is a function of the distance, r, between them:

$$f(r) = -\frac{A}{r^2} + \frac{B}{r^3} \qquad r > 0.$$

a. [2 points] Find the zeroes of f (in terms of A and B).

b. [7 points] Find the coordinates of the critical points and inflection points of f in terms of A and B.

c. [5 points] If f has a local minimum at $(1, -2)$ find the values of A and B. Using your values for A and B, justify that $(1, -2)$ is a local minimum.

8. [12 points] Below is a table of values for the function $t(y)$ which gives the number of tweets per day, in millions, on the social media website Twitter, y years after January 1, 2007. For this problem assume $t(y)$ is an increasing function.

year y	0	1	2	3	4
millions of tweets per day $t(y)$	0.005	0.3	2.5	35	50

a. [4 points] Using the table, estimate the expression

$$365\int_1^4 t(y)dy$$

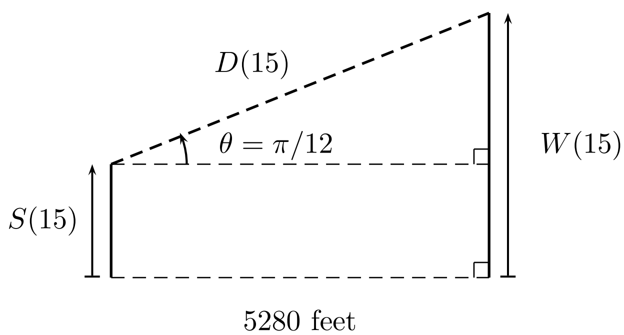
using a left-hand Riemann sum. Please write all of the terms in the sum for full credit.

b. [4 points] Give a practical interpretation of the expression $365\int_1^4 t(y)dy$.

c. [4 points] Suppose $T(y)$ is the total number of tweets, in millions, y years after January 1, 2007. If $T(3) = 9797$, estimate the total number of tweets between January 1, 2007 and January 1, 2011. Indicate what method you use to obtain your estimate and be sure to show your work.

9. [10 points] You are sitting on a ship traveling at a constant speed of 6 ft/sec, and you spot a white object due east moving parallel to the ship. After tracking the object through your telescope for 15 seconds, you identify it as the white whale. Let $W(t)$ denote the distance of the whale from its starting point in feet, and $S(t)$ denote the distance of the ship from its starting point in feet, with t the time in seconds since you saw the whale. You also note that at the end of the 15 seconds you were tracking the whale, you have turned your head $\pi/12$ radians north to keep it in your sights.

a. [1 point] If initially the creature is 5280 ft (1 mile) from the ship due east, use the angle you have turned your head to find the distance $D(t)$ in feet between the ship and the creature at 15 seconds. The figure below should help you visualize the situation. Recall that, for right triangles, $\cos(\theta)$ is the ratio of the adjacent side to the hypotenuse.



b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.