Math 115 — First Midterm

October 11, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	12	
3	12	
4	12	
5	6	
6	5	
7	10	
8	9	
9	12	
10	10	
Total	100	

1. [12 points] For each part below, give an explicit formula for a function which satisfies the given properties, if one exists. If such a function does not exist, explain why. Be sure to clearly indicate your final answer for each part.

a. [3 points] A continuous function, f, which is not differentiable.

Solution: The function $f(x) = |x|$ is continuous, but not differentiable. The function is continuous as it can be drawn without picking up one's pencil, but not differentiable because there is a corner on the graph at the point $(0, 0)$.

b. [3 points] A cubic polynomial, p, with two x-intercepts.

Solution: The function $p(x) = x^2(x-1) = x^3 - x^2$ is a cubic polynomial with x-intercepts at $x = 0, 1$.

c. [3 points] A continuous function, c, satisfying $\lim_{x\to 0^+} c(x) = -1$ and $\lim_{x\to 0^-} c(x) = 1$.

Solution: The function described here does not exist. If $\lim_{x\to 0^+} c(x) = -1$ and $\lim_{x\to 0^-} c(x) = 1$, then $\lim_{x\to 0} c(x)$ does not exist since the right and left hand limits are not equal. The function $c(x)$ is continuous at zero means the limit as $x \to 0$ exists and equals $c(0)$. If the right hand limit and the left hand limit are not the same, $\lim_{x\to 0} c(x)$ does not exist, and so $c(x)$ cannot be continuous at $x = 0$.

d. [3 points] A rational function, r, with a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 1$.

Solution: The function $r(x) = \frac{x}{x-1}$ has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 1$.

2. [12 points] The Facebook Data team has decided to track the University of Michigan network status updates that mention football in order to see which days to show ads for tailgating supplies. Starting at 1pm Saturday, they measure an aggregate Football Status Factor by calculating the percentage of status updates which mention any of a number of designated football terms every hour. They notice very quickly that the data is sinusoidal with period 168 hours (the number of hours in a week). Suppose $F(t)$ is this percentage, t hours after 1pm Saturday.

a. [2 points] If the maximum percentage is 96% at 1pm Saturday, and the minimum is 28% attained 84 hours later, compute the following quantities:

1. Midline

Solution: Since the maximum value is 96 and the minimum value is 28, the midline will be the average of these two values, that is $(96 + 28)/2 = 62$.

2. Amplitude

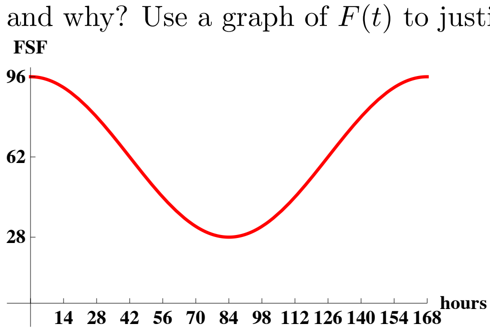
Solution: The amplitude of this sinusoidal function is half the distance between the maximum and minimum values, that is $(96 - 28)/2 = 34$. Notice that the midline plus the amplitude will be the maximum $64 + 32 = 96$ and the midline minus the amplitude will be the minimum $62 - 34 = 28$.

b. [6 points] Using the values computed above, find a formula for $F(t)$.

Solution: From the problem, we know that this function begins at its maximum, which makes cosine the natural choice for our sinusoidal function. We need the appropriate values of A,B, and k in $F(t) = A\cos(Bt) + k$. We are given that the period is 168 hours, so the value of $B = 2\pi/168$. We found the amplitude $A = 34$ and the midline $k = 62$ in part (a). So, the formula for $F(t)$ will be

$$F(t) = 34\cos\left(\frac{2\pi t}{168}\right) + 62.$$

c. [4 points] Suppose advertisers want to advertise when the rate at which people are talking about football is increasing the fastest. What time range would you recommend to them and why? Use a graph of $F(t)$ to justify your answer.



Solution: The Football Status factor gives an hourly rate of people talking about football since it averages mentions of football in Facebook statues over the course of an hour. To find when the rate at which people are talking about football is increasing the fastest, we just need to find when the Football Status Factor or $F(t)$ is increasing the fastest. The function F' will tell us whether $F(t)$ is increasing or decreasing, but to find maximum of F', we need to consider F'', the second derivative of F. At $t = 126$ hours, the graph is increasing and switches from concave up to concave down, so the derivative is at its largest when $t = 126$. Any time range around $t = 126$ would be the best time for advertisers to advertise.

3. [12 points] A zombie plague has broken out in Ann Arbor. As a nurse in the University of Michigan hospital, you saw the person with the first case of the plague, patient zero.

a. [2 points] In order to keep track of the growing zombie population in Ann Arbor, you collected the following data:

Days after patient zero	0	6	9	12
Number of Zombies	1	9	27	81

Would a linear function or an exponential function be the best model? Why?

Solution: In order for a function to be linear, it must have constant slope. Using the table above, we can compute the slopes between subsequent points: $(9-1)/6 = 4/3$, $(27-9)/(9-6) = 6$, and $(81-27)/(12-9) = 18$. Not only are these values not constant, they are increasing, so a linear function would be a very bad model.
If instead we take ratios of outputs with the same change in input we have $27/9 = 3 = 81/27$. Since this value is constant, an exponential function will be a good model for this data.

b. [4 points] Write a function $Z(t)$ of the appropriate type to model the growth of the zombie population with t measured in days after patient zero.

Solution: From part a), we know we are looking for an exponential function, which can take the form $Z(t) = Ab^t$ or $Z(t) = Ae^{kt}$. Since $Z(0) = 1$, the value of A for both types of exponential equation will be 1. Let's find $Z(t)$ of the form b^t. Since $Z(6) = 9$ and $Z(9) = 27$, we can take the ratio of these two equations:

$$\frac{27}{9} = \frac{b^9}{b^6} \Rightarrow 3 = b^{9-6} = b^3 \Rightarrow b = 3^{1/3} \approx 1.44225.$$

Therefore we can write the function exactly as $Z(t) = 3^{t/3}$, or approximate with $Z(t) = 1.44225^t$.
Similarly, we can solve for $Z(t)$ in the form e^{kt}:

$$\frac{27}{9} = \frac{e^{9k}}{e^{6k}} \Rightarrow 3 = e^{3k} \Rightarrow \ln(3) = 3k \Rightarrow k = \frac{\ln(3)}{3}.$$

Then we have $Z(t) = e^{\frac{\ln(3)t}{3}}$.

c. [3 points] The population of North America is approximately 530,000,000 people. Using your model, how long will it take until all but one person are infected?

Solution: To solve for the time it takes for all but one of 530,000,000 people to get infected, we need to set $Z(t) = 529999999$ and solve for t.

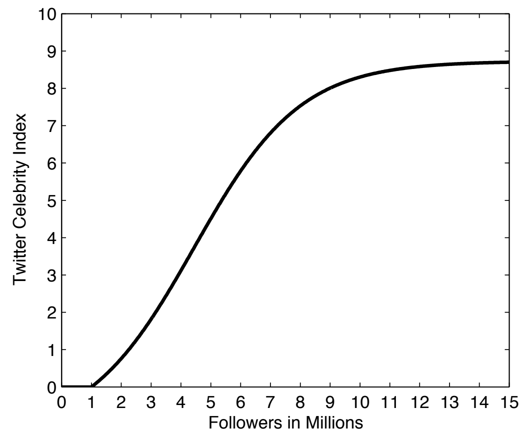
$$\begin{aligned} 3^{t/3} &= 529999999 \\ \frac{t}{3}\ln(3) &= \ln(529999999) \\ t &= \frac{3\ln(529999999)}{\ln(3)} \approx 54.856 \text{ days} \end{aligned}$$

d. [3 points] Using your table, approximate the instantaneous rate of change of the zombie population on the ninth day.

Solution: To approximate the instantaneous rate of change from the table, we need to compute the average rate of change either between the sixth day and the ninth day, the ninth day and the twelfth day, or the sixth day and the twelfth day.

$$\begin{aligned} \text{Average rate of change between 6th and 9th day} &= \frac{27-9}{9-6} = 6 \text{ zombies per day} \\ \text{Average rate of change between 9th and 12th day} &= \frac{81-27}{12-9} = 18 \text{ zombies per day} \\ \text{Average rate of change between 6th and 12th day} &= \frac{81-9}{12-6} = 12 \text{ zombies per day} \end{aligned}$$

4. [12 points] The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function $T(x)$ takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.



Use the graph above to help you answer the following questions.

a. [3 points] Explain in practical terms what $T(13.72) = 8.67$ means.

Solution: When a Twitter user has 13.72 million followers, their Twitter Celebrity index is 8.67.

b. [3 points] Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

Solution: When a user has a Twitter Celebrity index of 4.25, they have 4.88 million followers.

c. [3 points] Explain in practical terms what $T'(10) = 0.2278$ means.

Solution: When a Twitter user has 10 million followers, adding $100,000$ followers will increase their celebrity index by roughly .02278.

d. [3 points] Explain in practical terms what $(T^{-1})'(7.238) = 0.71$ means.

Solution: When a Twitter user has celebrity index 7.238, and increase of .1 to their index corresponds to gaining approximately .071 million (71,000) followers.

5. [6 points] Find a number k so that the following function is continuous on any interval.

$$j(t) = \begin{cases} (t+4)^3 & t < -2 \\ kt & t \geq -2 \end{cases}$$

Using your value of k, explain why this function is continuous on any interval.

Solution: On intervals not containing $t = -2$, this function is continuous since the functions $(t+4)^3$ and kt are both polynomials, and thus continuous, regardless of the value of k. So we must find the value of k which makes $j(t)$ continuous at $t = -2$. So we set

$$\lim_{t \to -2^-} j(t) = (-2+4)^3 = -2k = \lim_{t \to -2^+} j(t).$$

Solving this, we get $k = -4$. Now for any interval containing $t = -2$ we have that $j(t)$ is continuous.

6. [5 points] Using the limit definition of the derivative, write an explicit expression for the derivative of the function $E(x) = x^{\cos x}$ at $x = 2$. Do not try to calculate this derivative.

Solution:

$$E'(2) = \lim_{h \to 0} \frac{(2+h)^{\cos(2+h)} - 2^{\cos 2}}{h}.$$

7. [10 points] On the axes below sketch a well-labeled graph of a continuous function, g, which satisfies all of the following properties.

a. $g'(x) = 2$ for $1 < x < 2$

b. $g'(x) = -2$ for $2 < x < 3$

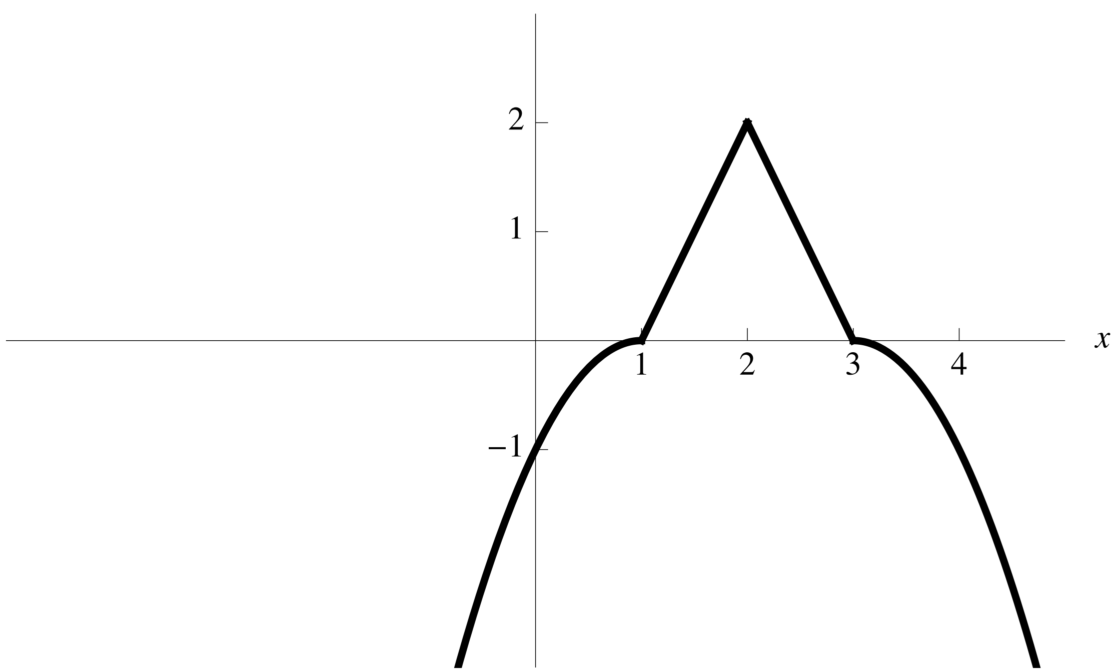
c. $g(0) = -1$

d. $g(1) = 0$

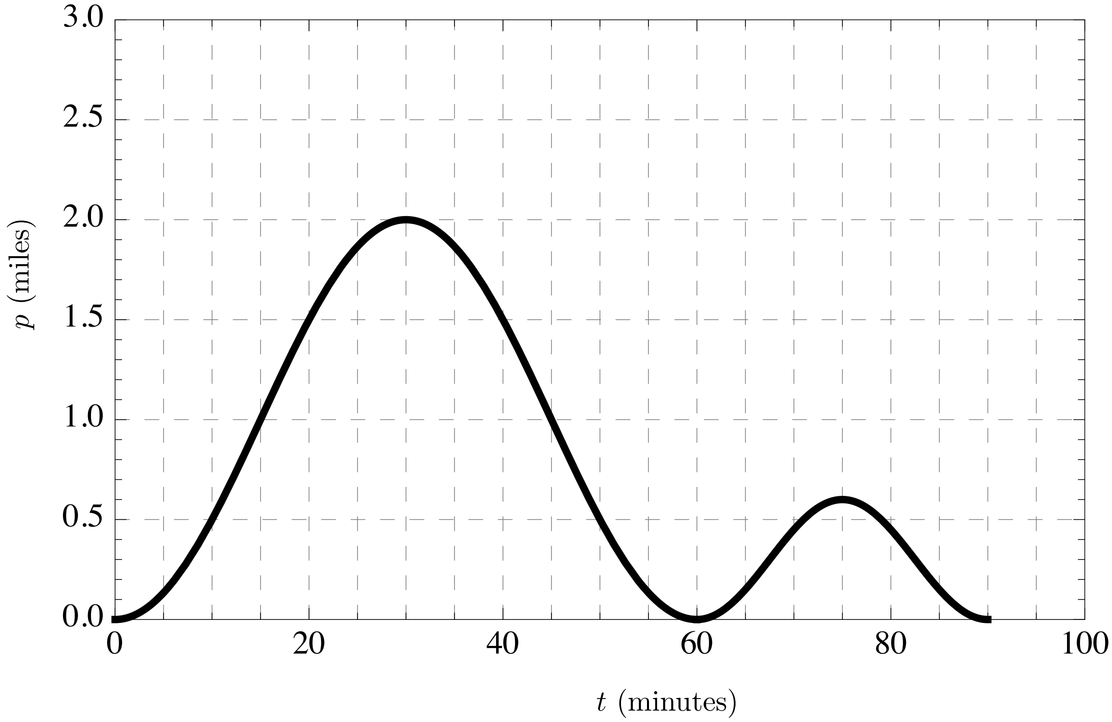
e. g is decreasing for $x > 3$

f. $g''(x) < 0$ for $x > 3$

g. g is concave down for $x < 1$



8. [9 points] The graph below shows a runner's distance, p, in miles from her starting point t minutes after she began to run.



Using the graph, estimate the following.

a. [3 points] All times during her run where her velocity was zero.

$t =$ $0, 30, 60, 75, 90$ **minutes**

b. [2 points] Her average velocity over the first 45 minutes of her run.

velocity= $1/45$ **miles per minute**

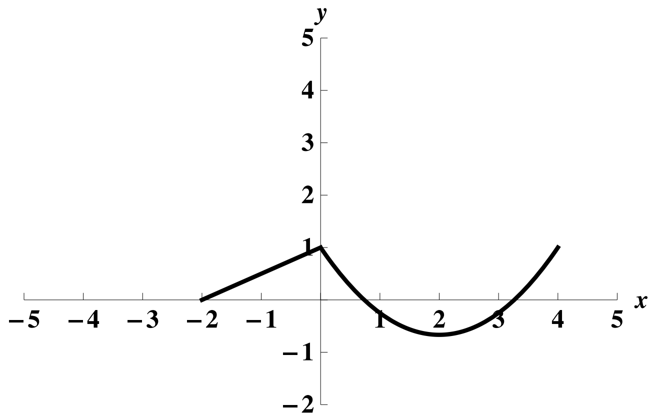
c. [2 points] Her average speed over the first 45 minutes of her run.

speed= $1/15$ **miles per minute**

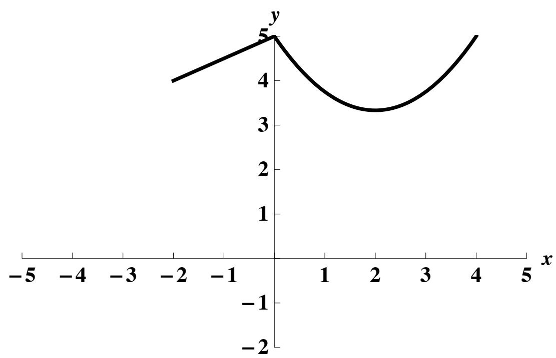
d. [2 points] Her velocity 80 minutes after she began running.

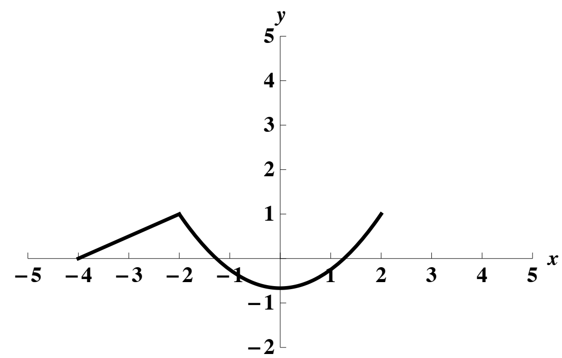
velocity= $-1/20$ **miles per minute**

9. [12 points] Consider the graph below of $g(x)$:

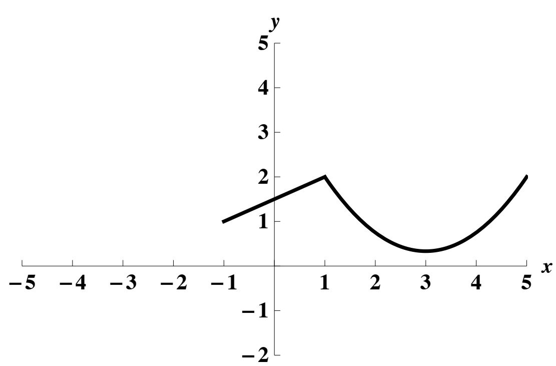


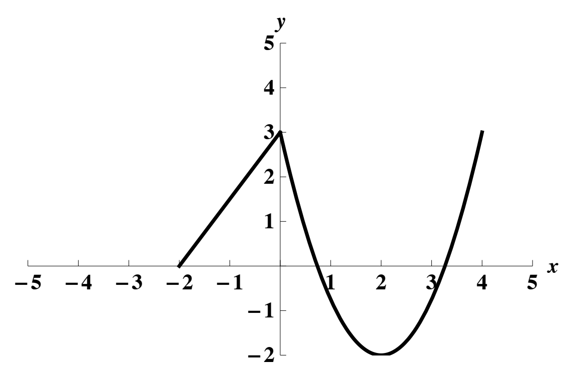
The four graphs below are shifts or stretches of $g(x)$. Write each function below in terms of $g(x)$.





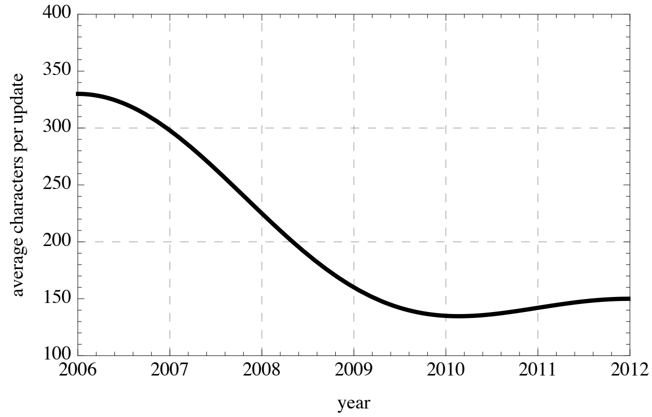
$h(x) =$ $g(x) + 4$ $k(x) =$ $g(x + 2)$





$\ell(x) =$ $g(x - 1) + 1$ $m(x) =$ $3g(x)$

10. [10 points] Facebook tracks the average number of characters used by its users to write their status updates. Below is the graph for a random (talkative) user from the beginning of 2006 to the beginning of 2012. Use the graph to answer the following questions.



a. [2 points] When were this user's status updates the longest? How long were they?

Solution: From the graph, the status updates are the longest when Facebook began tracking the length of status updates. The status updates are longest at the beginning of 2006 when they were an average of 330 characters.

b. [3 points] When was the length of the user's status updates decreasing? Increasing?

Solution: The status updates are decreasing from the beginning of 2006 until the beginning of 2010. The length of status updates is increasing from the beginning of 2010 to the beginning of 2012.

c. [2 points] When was the length of status updates shrinking the fastest?

Solution: The graph appears to have steepest negative slope around the beginning of 2008, so the length of status updates will be shrinking the fastest around then.

d. [3 points] Is this function continuous? Is it invertible? Justify your answer.

Solution: The function is continuous since there are no gaps in the function over the domain given. However, it is not invertible because it attains the same value at the beginning of 2012 and the beginning of 2009, so it fails the horizontal line test and is not invertible.