Math 115 — Second Midterm

November 15, 2011

Name: EXAM SOLUTIONS

Instructor: ____________ Section: ________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.
10. You must clearly indicate your final answer for each problem to receive credit.

Problem	Points	Score
1	9	
2	8	
3	12	
4	10	
5	8	
6	13	
7	16	
8	12	
9	12	
Total	100	

1. [9 points] Let $U = f(t)$ give the number of Facebook users in millions in year t. Suppose $f(2005) = 5.5$ and $f'(2005) = 4.9$. For this problem assume that $f(t)$ is strictly increasing.

a. [4 points] Find and interpret, in practical terms, $f^{-1}(5.5)$.

Solution:
The year in which there were 5.5 million Facebook users was 2005.

$f^{-1}(5.5) =$ 2005**(year)**

b. [5 points] Showing work, evaluate $(f^{-1})'(5.5)$. Interpret your answer in practical terms.

Solution:
$(f^{-1})'(5.5) = 1/(f'(f^{-1}(5.5)) = 1/f'(2005) = 1/4.9$.

When the number of Facebook users was 5.5 million, in 1/4.9 ($\sim$0.2) years the number of Facebook users will have increased by approximately 1 million users.

$(f^{-1})'(5.5) =$ 1/4.9 **years per million users**

2. [8 points] Recall the function $T(x)$ that took the number of followers (in millions) of a Twitter user and returned a value from 0 to 10 called the user's Twitter celebrity index. The derivative of $T(x)$ is given by the function

$$T'(x) = \frac{1532.5 \cdot (0.6)^x}{(5 + 60(0.6)^x)^2}.$$

a. [4 points] If $T(3) = 1.56$, compute the local linearization of $T(x)$ near $x = 3$.

Solution: The formula for the local linearization for a function $f(x)$ near a point $x = a$ is

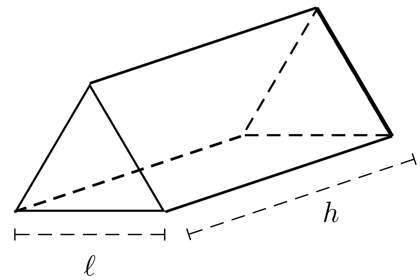
$$L(x) = f(a) + f'(a)(x - a).$$

To use this formula for $T(x)$, we will need to find $T'(3) = \frac{1532.5 \cdot (0.6)^3}{(5+60(0.6)^3)^2} \approx 1.0262$. Then, the local linearization is $L(x) = 1.56 + 1.0262(x - 3)$.

b. [4 points] Use your expression from (a) to approximate the Twitter celebrity index of a celebrity with 3.2 million followers.

Solution: $L(3.2) = 1.56 + 1.0262(3.2 - 3) = 1.7652$ (TCI).

3. [12 points] Consider the prism with equilateral triangles of side length ℓ centimeters for ends and a length of h centimeters, illustrated below. The volume of this prism is $\sqrt{3}\,\ell^2 h/4$. You may find it useful to note that the area of an equilateral triangle of side length ℓ is $\sqrt{3}\,\ell^2/4$.



a. [4 points] Give the equation of the surface area of this prism, listing units.

Solution: To find the surface area, we must add up the area of all the faces, the two ends and the three rectangular sides. Therefore, Surface area $= \sqrt{3}/2\ell^2 + 3\ell h$ square centimeters.

Surface area= $\sqrt{3}/2\ell^2 + 3\ell h$ **cm²**

b. [8 points] If the prism has a fixed volume of 16 cm^3, find the values of ℓ and h which minimize the surface area. Clearly justify that you have found the minimum.

Solution: We can proceed in one of two ways; we can either use the formula for volume to solve for h in terms of ℓ or ℓ in terms of h. Since volume is linear in h, it is easier to solve for h in terms of ℓ as $h = 64/\sqrt{3}\ell^{-2}$. When we plug this into the surface area equation, we get

$$S(\ell) = \frac{\sqrt{3}\,\ell^2}{2} + \frac{3(64)}{\ell\sqrt{3}}.$$

To find the minimum, we first compute the critical points of this equation:

$$\begin{aligned} S'(\ell) &= \sqrt{3}\,\ell - \frac{\sqrt{3}(64)}{\ell^2} \\ S'(\ell) = 0 \quad \Rightarrow& \quad \sqrt{3}\,\ell - \frac{\sqrt{3}(64)}{\ell^2} = 0 \\ & \quad \sqrt{3}\,\ell = \frac{\sqrt{3}(64)}{\ell^2} \\ & \quad \ell^3 = 64 \Rightarrow \ell = 4. \end{aligned}$$

To see that this critical point is a minimum, consider the second derivative. Since $S''(\ell) = \sqrt{3} + \frac{2\sqrt{3}(64)}{\ell^3} > 0$ for all all positive ℓ, the function $S(\ell)$ is concave up, and thus $\ell = 4$ is the local and global minimum. Therefore, the dimension of the equilateral prism with volume 16 cm^3 and minimal surface area is $\ell = 4$ cm and $h = 4/\sqrt{3}$ cm

4. [10 points] The cable of a suspension bridge with two supports $2L$ meters apart hangs H meters above the ground. The height H is given in terms of the distance in meters from the first support x (in meters) by the function

$$H(x) = e^{x-L} + e^{L-x} + H_0 - 2$$

where H_0 and L are positive constants. Notice that x ranges from 0 (the first support) to $2L$ (the second support).

a. [4 points] Find (but do not classify) the critical points for the function $H(x)$.

Solution: To find the critical points, we first take the derivative of $H(x)$:

$$H'(x) = e^{x-L} - e^{L-x}.$$

To find the critical points, we set $H'(x) = 0$:

$$\begin{aligned} H'(x) = 0 \quad &\Rightarrow \quad e^{x-L} - e^{L-x} = 0 \Rightarrow e^{x-L} = e^{L-x} \\ \text{apply } \ln \text{to both sides} \quad & \quad x - L = L - x \\ & \quad 2x = 2L \Rightarrow x = L \end{aligned}$$

So, $H(x)$ has only one critical point at $x = L$.

b. [6 points] Find the x and y coordinates of all global maxima and minima for the function $H(x)$. Justify your answers.

Solution: Since the values of x lie in the closed interval $[0, 2L]$, to find all global maxima and minima, we need to compare the values of H at the endpoints and at any critical points. From part (a), we know the only critical point is at $x = L$, we plug $x = 0, L$, and $2L$ into $H(x)$:

$H(0) = e^{-L} + e^{L} + H_0 - 2$, $H(L) = 1 + 1 + H_0 - 2 = H_0$, and $H(2L) = e^{L} + e^{-L} + H_0 - 2$.

To identify which of these should be larger, we notice that $e^x + e^{-x} \geq 2$ for all x. Therefore, $H(2L) = H(0) > H(L)$. Then, the function $H(x)$ has global maxima at $(0, e^{-L} + e^{L} + H_0 - 2)$ and $(2L, e^{-L} + e^{L} + H_0 - 2)$ and a global minimum at (L, H_0).

5. [8 points] Each part of this problem has four statements, (i)-(iv). For each part, circle all statements which are always true and draw a line through all other statements. Any ambiguous markings will receive no credit.

a. [4 points] Let $q(t) = A\cos(Bt) + C\sin(Bt)$, with A, B, and C constants.

(i) $\boxed{q''(t) = -B^2 q(t).}$

(ii) ~~The function $q(t)$ is concave down everywhere.~~

(iii) ~~The value of $q'\left(\frac{\pi}{2B}\right)$ is AB.~~

(iv) $\boxed{\text{If } q'(0) = \pi \text{ and } C = 2, \text{ then } q(t) = q(t+4) \text{ for all values of t.}}$

b. [4 points] Let $f(x)$ be a function defined on the closed interval $[0, 4]$, such that $f''(x) > 0$ on the entire interval, and $f'(x)$ is zero only at $x = 3$.

(i) ~~$f(1) > f(4)$.~~

(ii) $\boxed{f'(1) < f'(3).}$

(iii) ~~The point $(3, f(3))$ is a local maximum.~~

(iv) $\boxed{\text{Either one or both of } f(4) \text{ and } f(0) \text{ are a global maximum.}}$

6. [13 points] Let $f(v)$ be the gas consumption (in liters/km) of a car going at velocity v (in km/hr). In other words, $f(v)$ tells you how many liters of gas the car uses to go one kilometer, if it is going at velocity v. You are told that

$$f(90) = 0.08 \text{ and } f'(90) = 0.0008.$$

a. [5 points] Let $g(v)$ be the distance the same car goes on one liter of gas at velocity v. What is the relationship between $f(v)$ and $g(v)$? Find $g(90)$ and $g'(90)$.

Solution: The function $f(v)$ is consumption in L/km and $g(v)$ is efficiency in km/L, so the relationship between $f(v)$ and $g(v)$ is

$$g(v) = 1/f(v) = [f(v)]^{-1}.$$

This means $g(90) = 1/f(90) = 1/0.08 = 12.5$ km/L. The derivative of $g(v)$ is

$$g'(v) = -[f(v)]^{-2}f'(v).$$

so $g'(90) = -[f(90)]^{-2}f'(90) = -(0.08)^{-2}(0.0008) = -0.125$ km/L per km/h.

b. [5 points] Let $h(v)$ be the gas consumption in liters per hour. In other words, $h(v)$ tells you how many liters of gas the car uses in one hour if the car is going at velocity v. What is the relationship between $h(v)$ and $f(v)$? Find $h(90)$ and $h'(90)$.

Solution: Since $f(v)$ is consumption in L/km and v is velocity in km/h, the function $h(v)$ must be the product of v and $f(v)$, in L/h.

$$h(v) = vf(v).$$

This means $h(90) = 90f(90) = 7.2$ L/h. The derivative of $h(v)$ is

$$h'(v) = f(v) + vf'(v)$$

so $h'(90) = f(90) + 90f'(90) = 0.152$ L/h per km/h.

c. [3 points] How would you explain the practical meaning of $g'(90)$ to a driver who knows no calculus?

Solution: The value of $g'(90)$ is -0.125 km/L per km/h. In practical terms this means: "When the car increases speed from 90 to 91 km/h, the fuel efficiency of the car decreases by about 0.125 km/L."

7. [16 points] Let $f(x) = \ln(x)$. Use the table of values below for $g(x)$ and $g'(x)$ to answer the following questions.

x	2	3	4
$g(x)$	1	4	6
$g'(x)$	5	3	2

a. [4 points] If $F(x) = f(g(x))$, find $F'(4)$.

Solution:

$$F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x))g'(x) \text{ by the Chain Rule}$$

Then, to find $F'(4)$, we plug 4 in for x and take the correct values from the table:

$$F'(4) = f'(g(4))g'(4) = \frac{g'(4)}{g(4)} = \frac{2}{6} = \frac{1}{3}.$$

b. [4 points] If $G(x) = g^{-1}(x)$, find $G'(4)$.

Solution:

$$G(x) = g^{-1}(x) \quad \Rightarrow \quad g(g^{-1}(x)) = x \text{ and by the Chain Rule } \left(g^{-1}\right)'(x) = \frac{1}{g'(g^{-1}(x))}$$

Then, to find $G'(4)$, we plug 4 in for x and take the correct values from the table:

$$G'(4) = \frac{1}{g'(g^{-1}(4))} = \frac{1}{g'(3)} = \frac{1}{3}.$$

c. [4 points] If $H(x) = \tan(g(x))$, find $H'(3)$.

Solution: To find the derivative of H, you can either recall the derivative of tangent from your notecard, or use the quotient rule.

$$H(x) = \tan(g(x)) \quad \Rightarrow \quad H'(x) = \frac{g'(x)}{\cos^2(g(x))}.$$

Then, to find $H'(3)$, we plug 3 in for x and take the correct values from the table:

$$H'(3) = \frac{g'(3)}{\cos^2(g(3))} = \frac{3}{\cos^2(4)} \approx 7.0217.$$

d. [4 points] If $E(x) = e^{f(x)g(x)}$, find $E'(2)$.

Solution:

$$E(x) = e^{f(x)g(x)} \quad \Rightarrow \quad E'(x) = e^{f(x)g(x)} \left(f'(x)g(x) + f(x)g'(x)\right)$$

$$\text{Since } f(x) = \ln(x), \qquad E'(z) = e^{\ln(x)g(x)} \left(\frac{g(x)}{x} + \ln(x)g'(x)\right).$$

Then, to find $E'(2)$, we plug 2 in for x and take the correct values from the table:

$$E'(2) = e^{\ln(2)g(2)} \left(\frac{g(2)}{2} + \ln(2)g'(2)\right) = e^{\ln(2)} \left(\frac{1}{2} + 5\ln(2)\right) \approx 7.9315.$$

8. [12 points] The equation $(x^2 + y^2)^2 = 4x^2y$ describes a two-petaled rose curve.

a. [2 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At the point $(x, y) = (1, 1)$,

$$(x^2 + y^2)^2 = (1^2 + 1^2)^2 = 4 = 4(1)^2(1) = 4x^2y.$$

b. [7 points] Calculate dy/dx at $(x, y) = (1, 1)$.

Solution: Differentiating both sides of the equation for the curve with respect to x we have

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 4\left(2xy + x^2\frac{dy}{dx}\right).$$

At the point $(x, y) = (1, 1)$ this equation becomes

$$2(1^2 + 1^2)\left(2(1) + 2(1)\frac{dy}{dx}\right) = 4\left(2(1)(1) + (1)^2\frac{dy}{dx}\right).$$

Simplifying, we have $4(2 + 2\frac{dy}{dx}) = 8 + 4\frac{dy}{dx}$. This gives us that $\frac{dy}{dx} = 0$ at $(x, y) = (1, 1)$.

c. [3 points] Find the equation of the tangent line to the rose curve at the point $(x, y) = (1, 1)$.

Solution: Using point slope form, the tangent line is $y - 1 = 0(x - 1)$. Simplifying, we have that the tangent line to the rose curve at $(x, y) = (1, 1)$ is $y = 1$.

9. [12 points] Suppose $w(x)$ is an everywhere differentiable function which satisfies the following conditions:

- $w'(0) = 0$.
- $w'(x) > 0$ for $x > 0$.
- $w'(x) < 0$ for $x < 0$.

Let $f(t) = t^2 + bt + c$ where b and c are positive constants with $b^2 > 4c$. Define $L(t) = w(f(t))$.

a. [2 points] Compute $L'(t)$. Your answer may involve w and/or w' and constants b and c.

Solution: $L'(t) = w'(t^2 + bt + c) \cdot (2t + b)$.

b. [4 points] Using your answer from (a), find the critical points of $L(t)$ in terms of the constants b and c.

Solution: $L(t)$ has critical points when $L'(t) = 0$. This happens only if $w'(t^2+bt+c) = 0$ or if $(2t + b) = 0$.
$w'(t^2 + bt + c) = 0$ means $t^2 + bt + c = 0$ by the first property of w' above. Solving using the quadratic formula, we have

$$t = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

as critical points of $L(t)$. Both of these roots exist and are distinct since $b^2 > 4c$.
If $2t + b = 0$, we have $t = -\frac{b}{2}$ as a critical point. Altogether our critical points are $t = -\frac{b}{2}, -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}, -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

c. [6 points] Classify each critical point you found in (b). Be sure to fully justify your answer.

Solution: For simplicity, let's set $p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ and $m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$.

We know that $f(t)$ is an upward opening parabola with roots at p and m. We also know $p > m$, so this means $f(t) > 0$ for $t < m$ and $t > p$. This also means $f(t) < 0$ for $m < t < p$. Thus by properties two and three of w' above we know $w'(f(t)) > 0$ for $t < m$ and $t > p$, and $w'(f(t)) < 0$ for $m < t < p$.

The expression $2t + b$ is positive for $t > -\frac{b}{2}$ and negative for $t < -\frac{b}{2}$.

Putting all of this information together gives us

$$L'(t) > 0$$

on the intervals $(m, -\frac{b}{2})$ and $(p, +\infty)$, and

$$L'(t) < 0$$

on the intervals $(-\infty, m)$ and $(-\frac{b}{2}, p)$. Thus, by the first derivative test, the critical points $t = m = -\frac{b}{2} - \frac{\sqrt{b^2-4c}}{2}$ and $t = p = -\frac{b}{2} + \frac{\sqrt{b^2-4c}}{2}$ are local minima, and $t = -\frac{b}{2}$ is a local maximum.