Math 115 — Second Midterm

November 12, 2013

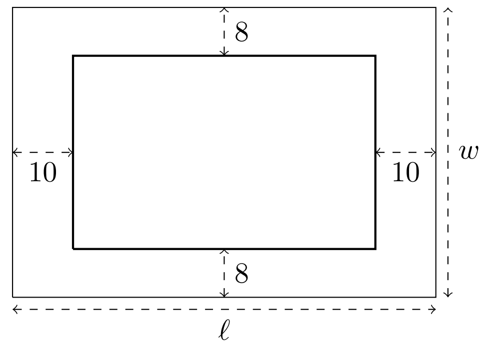
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	7	
2	6	
3	9	
4	17	
5	6	
6	8	
7	14	
8	12	
9	13	
10	8	
Total	6	

1. [7 points] Liam wants to build a rectangular swimming pool behind his new house. The pool will have an area of 1600 square feet. He will have 8-foot wide decks on two sides of the pool and 10-foot wide decks on the other two sides of the pool (see the diagram below).



a. [4 points] Let ℓ and w be the length and width (in feet) of the pool area including the decks as shown in the diagram. Write a formula for ℓ in terms of w.

Solution: The area of the pool needs to be 1600 sq ft, so

$$(\ell - 2(10))(w - 2(8)) = 1600$$

Solving this for ℓ gives

$$\ell = \frac{1600}{w-16} + 20$$

b. [3 points] Write a formula for the function $A(w)$ which gives the total area (in square feet) of the pool **and** the decks in terms of only the width w. Your formula should not include the variable ℓ. (This is the function Liam would minimize in order to find the minimum area that his pool and deck will take up in his yard. You do not need to do the optimization in this case.)

Solution: The pool and decks together make a rectangle of length ℓ and width w. The area A of the rectangle is $A = \ell w$. Substituting the formula from part (a) gives

$$A(w) = \left(\frac{1600}{w-16} + 20\right) w$$

2. [6 points] Given the implicit curve $y^2 = \cos(xy) - 3x$, find $\dfrac{dy}{dx}$.

Solution: Using implicit differentiation, we get

$$2y\frac{dy}{dx} = -\sin(xy)\left(y + x\frac{dy}{dx}\right) - 3$$

Solving for $\dfrac{dy}{dx}$ gives

$$\begin{aligned} 2y\frac{dy}{dx} &= -y\sin(xy) - x\sin(xy)\frac{dy}{dx} - 3 \\ 2y\frac{dy}{dx} + x\sin(xy)\frac{dy}{dx} &= -y\sin(xy) - 3 \\ \frac{dy}{dx} &= \frac{-y\sin(xy) - 3}{2y + x\sin(xy)} \end{aligned}$$

3. [9 points] This problem concerns the function $f(x) = -x - 3e^{4x}$.

a. [3 points] Show that the function f is invertible.

Solution: We have $f'(x) = -1 - 12e^{-4x}$ which is negative for all values of x. This means that f is a strictly decreasing function. Since f is strictly decreasing, it never takes the same value twice so f is invertible.

b. [2 points] Find $f^{-1}(-3)$. You do not need to show any work.

Solution: $f^{-1}(-3) = 0$ because $f(0) = -3$.

c. [4 points] Evaluate $(f^{-1})'(-3)$. Show all of your work.

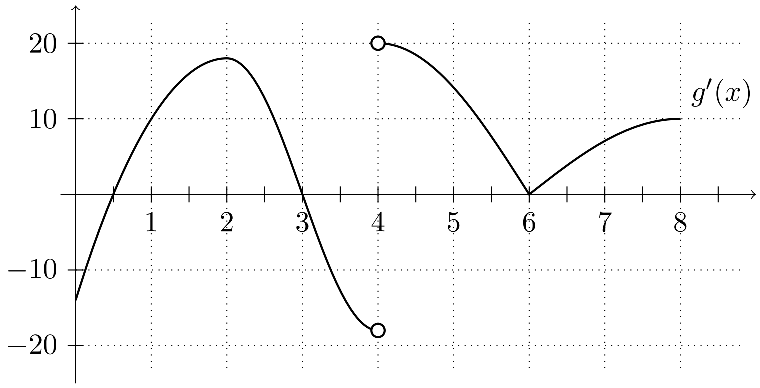
Solution: Using the formula for the derivative of an inverse function, we get

$$(f^{-1})'(-3) = \frac{1}{f'(f^{-1}(-3))} = \frac{1}{f'(0)}$$

Since $f'(x) = -1 - 12e^{-4x}$ we have $f'(0) = -13$ and so

$$(f^{-1})'(-3) = -\frac{1}{13}$$

4. [17 points] The function $g(x)$ is continuous on the interval $0 < x < 8$. The graph of $g'(x)$, the **derivative** of $g(x)$, is shown below.



a. [6 points] List the x-coordinates of the critical points of the function $g(x)$ and state whether each is a local maximum, local minimum, or neither. You do not need to justify your answers.

Solution:	$x = 0.5$	$x = 3$	$x = 4$	$x = 6$
	local minimum	local maximum	local minimum	neither

b. [3 points] List the x-coordinates of the inflection points of the function $g(x)$. You do not need to justify your answers.

Solution: $x = 2$, $x = 6$

c. [3 points] Suppose that $g(1) = 8$. Write an equation for the best linear approximation to $g(x)$ at $x = 1$.

$g(x) \approx$ $10(x-1)+8$

d. [2 points] Use your approximation from part **(c)** to estimate $g(1.05)$.

Solution: $g(1.05) \approx 10(1.05 - 1) + 8 = 8.5$

e. [3 points] Is your estimate for $g(1.05)$ an overestimate or an underestimate? Explain.

Solution: We see from the graph that $g'(x)$ is increasing at $x = 1$, so $g(x)$ is concave up at $x = 1$. Because the graph of $g(x)$ is concave up at $x = 1$, the tangent line is below the curve so our estimate is an underestimate.

5. [6 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

a. [2 points] If the function $f(x)$ is continuous on the interval $(0, 100)$, then $f(x)$ has a global maximum and a global minimum on that interval.

True $\boxed{\text{False}}$

b. [2 points] If $f(x)$ is a differentiable function with a critical point at $x = c$, then the function $g(x) = e^{f(x)}$ also has a critical point at $x = c$.

$\boxed{\text{True}}$ False

c. [2 points] If $f'(x)$ is continuous and $f'(x) \neq 0$ for all x, then $f(0) \neq f(5)$.

$\boxed{\text{True}}$ False

6. [8 points] This problem concerns the implicit curve

$$x^2 + xy + y^2 = 7$$

for which

$$\frac{dy}{dx} = \frac{-y - 2x}{x + 2y}.$$

a. [3 points] Find an equation for the tangent line to the curve at the point $(1, 2)$.

Solution:

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-2 - 2(1)}{1 + 2(2)} = -\frac{4}{5}$$

So the tangent line at $(1, 2)$ is $y = -\frac{4}{5}(x - 1) + 2$.

b. [5 points] Find the x- and y-coordinates of all points on the curve at which the tangent line is vertical.

Solution: If the tangent line is vertical, the slope will be undefined. The derivative $\frac{dy}{dx}$ is undefined when $x + 2y = 0$ which means $x = -2y$. Plugging this into the equation for the curve, we get

$$\begin{aligned} (-2y)^2 + (-2y)y + y^2 &= 7 \\ y^2 &= \frac{7}{3} \\ y &= \pm\sqrt{\frac{7}{3}} \end{aligned}$$

Since $x = -2y$ this gives two points $\left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right)$ and $\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right)$.

7. [14 points] The table of values below gives information about the first and second derivatives of a function $f(x)$.

x	-3	-2	-1	0	1	2	3
$f'(x)$	-2	0	-1	0	2	0	-2
$f''(x)$	2	0	0	0	0	-2	-1

Assume that $f''(x)$ is **continuous** on $[-3, 3]$ and that the values of $f'(x)$ and $f''(x)$ are either **strictly positive** or **strictly negative** between consecutive table entries. You do not need to show work or give an explanation for this problem, but any unclear answers will be marked as incorrect.

a. [4 points] On which of the following intervals is $f''(x) < 0$? Circle ALL correct answers.

$-3 < x < -2$ $\boxed{-2 < x < -1}$ $-1 < x < 0$ $0 < x < 1$ $\boxed{1 < x < 2}$ $\boxed{2 < x < 3}$

b. [10 points] For each of the following x values, circle ALL answers that apply. If none of the choices apply, don't circle anything.

At $x = -2$, f has a **local maximum** **local minimum** $\boxed{\textbf{inflection point}}$

At $x = -1$, f has a **local maximum** **local minimum** $\boxed{\textbf{inflection point}}$

At $x = 0$, f has a **local maximum** $\boxed{\textbf{local minimum}}$ **inflection point**

At $x = 1$, f has a **local maximum** **local minimum** $\boxed{\textbf{inflection point}}$

At $x = 2$, f has a $\boxed{\textbf{local maximum}}$ **local minimum** **inflection point**

8. [12 points] For Thanksgiving, Bert is trying to make a festive feast table using fall-colored cloth and other accessories. The cloth costs \$0.25 per square foot and the accessories are \$0.50 each. He decides the impact of the festive table, I, is a function of the number of square feet of cloth, c, that he uses and the number of accessories, a, that he uses. This relationship is given by

$$I = c\left(\frac{1}{2}a - 3\right)^2.$$

Bert has a total budget of \$9 for the cloth and accessories.

a. [2 points] Write an equation which expresses that the total cost of the cloth plus the accessories for the festive table is \$9.

Solution:

$$0.25c + 0.5a = 9$$

b. [10 points] Use your answer from **(a)** to find the maximum impact of the festive table that is possible for \$9, as well as how many accessories and how much cloth is needed to achieve the maximum impact. Be sure to show your answer is indeed the maximum.

Solution: From part (a) we get

$$a = 18 - 0.5c$$

Plugging this into the formula for impact gives

$$I = c\left(\frac{1}{2}(18 - 0.5c) - 3\right)^2 = c\left(6 - \frac{1}{4}c\right)^2$$

We need to maximize I on the domain $0 \le c \le 36$. Taking the derivative with respect to c gives

$$\frac{dI}{dc} = \left(6 - \frac{1}{4}c\right)^2 + c\left(2\left(6 - \frac{1}{4}c\right)\left(-\frac{1}{4}\right)\right) = \left(6 - \frac{1}{4}c\right)\left(6 - \frac{3}{4}c\right)$$

Then $\dfrac{dI}{dc} = 0$ when $c = 24$ or $c = 8$. We test the critical points and the endpoints:

$$\begin{array}{ll} c = 0 & I = 0 \\ c = 8 & I = 128 \\ c = 24 & I = 0 \\ c = 36 & I = 324 \end{array}$$

and find the the maximum impact $I = 324$ occurs if $c = 36$. Using $a = 18 - 0.5c$ we find that $a = 0$ at this point.

maximum impact: $I =$ **324**

$c =$ **36**

$a =$ **0**

9. [13 points] Consider the function

$$f(x) = ax \ln x - bx$$

with domain $x > 0$, where a and b are positive constants. Note that this function has exactly one critical point.

a. [3 points] Find $f'(x)$.

Solution:

$$f'(x) = a\left((1)\ln x + x\frac{1}{x}\right) - b = a\ln x + a - b$$

b. [4 points] For which values of a and b does $f(x)$ have a critical point at $(e, -2)$?

Solution: We want $f'(e) = 0$ and $f(e) = -2$. We plug $x = e$ into $f'(x)$ to get

$$f'(e) = a\ln e + a - b = 2a - b = 0$$

which tells us that $2a = b$. Plugging $x = e$ into $f(x)$ gives

$$f(e) = ae\ln e - be = (a - b)e = -2.$$

Using $2a = b$, we get

$$(a - (2a))e = -2$$

so that $a = 2/e$. Since $b = 2a$, we get $b = 4/e$.

c. [3 points] Using your values of a and b from part **(b)**, is the critical point from **(b)** a local maximum, local minimum, or neither? Justify your answer.

Solution: The second derivative of $f(x)$ is $f''(x) = a/x = (4/e)/x$. Then $f''(e) = 4/e^2 > 0$ so the graph of $f(x)$ is concave up at $x = e$. This means that $f(x)$ has a local minimum at $x = e$.

d. [3 points] Using your values of a and b from part **(b)**, find the x-coordinates of any inflection points of $f(x)$ or show that $f(x)$ has no inflection points.

Solution: The second derivative of $f(x)$ is $f''(x) = a/x = (4/e)/x$. This is continuous and positive for all values of x in the domain $x > 0$ of $f(x)$. Since the second derivative never changes sign, $f(x)$ has no inflection points.

10. [8 points] For each question below, circle the answer that correctly completes the statement. There is exactly one correct answer per problem. You do not need to show any work or give any explanation. There is no penalty for guessing. Any unclear marks will receive no credit.

a. [2 points] Suppose $f(x) = x^4 - 2a^2x^2 + 2a^2$ where $a > 2$ is a positive constant. The critical points of $f(x)$ are at $x = 0, \pm a$. Then the global maximum of $f(x)$ on $[-2a, 1]$ occurs at

$x = a$ $\quad$ $x = -a$ $\quad$ $x = \pm a$ $\quad$ $x = 0$ $\quad$ $\boxed{x = -2a}$ $\quad$ $x = 1$

b. [2 points] Suppose $f(x) = x^4 - 2a^2x^2 + 2a^2$ where $a > 2$ is a positive constant. The critical points of $f(x)$ are at $x = 0, \pm a$. Then the global minimum of $f(x)$ on $[-2a, 1]$ occurs at

$x = a$ $\quad$ $\boxed{x = -a}$ $\quad$ $x = \pm a$ $\quad$ $x = 0$ $\quad$ $x = -2a$ $\quad$ $x = 1$

c. [2 points] If $g(x)$ is a positive differentiable function, then for $x > 0$, the derivative of the function $\dfrac{\ln x}{g(x)}$ is

$\dfrac{1}{xg(x)}$ $\qquad$ $\dfrac{1}{xg'(x)}$ $\qquad$ $\boxed{\dfrac{1}{xg(x)} - \dfrac{g'(x)\ln x}{(g(x))^2}}$

$\dfrac{1}{xg(x)} + \dfrac{g'(x)\ln x}{(g(x))^2}$ $\qquad$ $\dfrac{1}{xg'(x)} - \dfrac{\ln x}{g(x)^2}$ $\qquad$ $\dfrac{1}{xg'(x)} + \dfrac{\ln x}{g(x)^2}$

d. [2 points] Suppose the local linearization of a function $h(x)$ at the point $(x, y) = (2, -1)$ gives the estimate $h(2.1) \approx -0.88$. The value of $h'(2)$ is

0.12 $\quad$ -0.12 $\quad$ 8.8 $\quad$ -8.8 $\quad$ $\boxed{1.2}$ $\quad$ -1.2