Math 115 — Final Exam

December 17, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	12	
3	12	
4	10	
5	12	
6	12	
7	10	
8	11	
9	10	
Total	100	

1. [11 points] At a recent UM football game, a football scientist was measuring the excitement density, $E(x)$, in cheers per foot, in a one hundred foot row of the football stadium where x is the distance in feet from the beginning of the row. He took measurements every twenty feet and the data is recorded in this table.

x	0	20	40	60	80	100
$E(x)$	30	24	19	16	13	7

Assume for this problem that $E(x)$ is a decreasing function for $0 \le x \le 100$.

a. [6 points] Write a right sum and a left sum which approximate the total cheers in the row. Be sure to write all of the terms for each sum.

Solution:
LEFT $= 20(30) + 20(24) + 20(19) + 20(16) + 20(13) = 2040$
RIGHT $= 20(24) + 20(19) + 20(16) + 20(13) + 20(7) = 1580$

b. [2 points] Indicate whether the right and left sums are overestimates or underestimates for the total number of cheers in the row.

The right sum is an **overestimate** $\boxed{\textbf{underestimate}}$

The left sum is an $\boxed{\textbf{overestimate}}$ **underestimate**

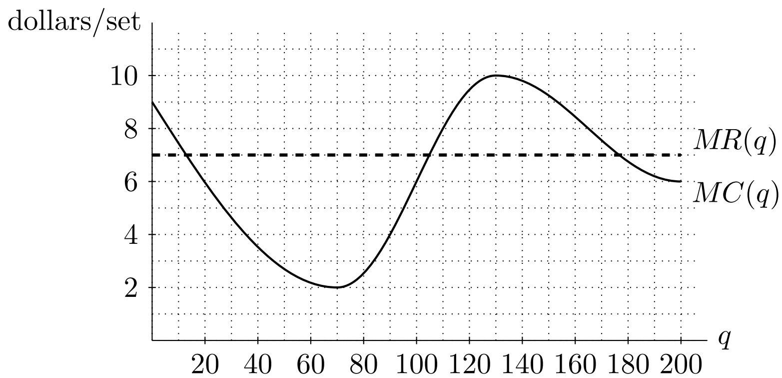
c. [3 points] How many measurements must the scientist take to guarantee that the left sum approximates the total number of cheers in the row within 5 cheers of the actual number?

Solution: The actual number of cheers is somewhere in between the left and right estimates because $E(x)$ is decreasing. So we want the difference between the left and right sums to be less than or equal to 5. If n is the number of subintervals used in the estimates, we have

$$|\text{Left} - \text{Right}| = |E(0) - E(100)| \cdot \frac{100-0}{n} = \frac{2300}{n}$$

We need $\frac{2300}{n} \le 5$, which is true if $n \ge 460$. We need 460 subintervals in our left sum estimate, so we need at least 460 measurements.

2. [12 points] Link has started a business selling winter clothes for cats. Among his most successful products are his new kitten mittens. He is currently selling his mittens for \$7 per set. Below is a graph of Link's marginal cost $MC(q)$ and marginal revenue $MR(q)$, in dollars per set of mittens, if he makes q sets of mittens this winter. Due to a shortage of yarn, Link can make a maximum of 200 sets of mittens this winter. In order to start making mittens, Link must spend \$40 on knitting supplies (in other words, it costs \$40 to make 0 sets of mittens).



You do not need to show any work for this problem.

a. [3 points] Approximately how many sets of mittens should Link make this winter in order to maximize his profit?

Answer: Link should make about **104** sets of mittens.

b. [2 points] If the price per set is raised to \$9, approximately how many sets of mittens should Link make in order to maximize his profit?

Answer: Link should make about **200** sets of mittens.

c. [3 points] Write an expression involving integrals which equals Link's total profit if Link makes 150 sets of mittens. Your expression may involve the functions $MR(q)$ and $MC(q)$.

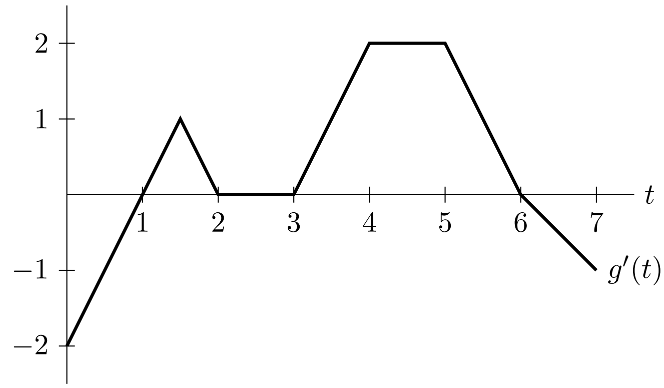
Solution:

$$\int_0^{150} (MR(q) - MC(q))\, dq - 40$$

d. [4 points] Link makes a deal with a store that would like to buy his cat hats. If the store buys up to 50 hats, then each one will cost \$10. If the store buys more than 50 hats, then Link will reduce the price of the entire order by \$0.05 per hat for every additional hat over 50. (For example, if the store buys 52 hats, they will pay \$9.90 per hat.) Write a formula for a function $L(q)$ which gives Link's revenue if he sells q hats to the store.

$$L(q) = \begin{cases} 10q & \text{if } 0 \le q \le 50 \\ (10 - 0.05(q-50))q & \text{if } q > 50 \end{cases}$$

3. [12 points] The function $g(t)$ is the volume of water in the town water tank, in thousands of gallons, t hours after 8 A.M. A graph of $g'(t)$, the **derivative** of $g(t)$, is shown below. Note that $g'(t)$ is a piecewise-linear function.



a. [4 points] Write an integral which represents the average rate of change, in thousands of gallons per hour, of the volume of water in the tank between 9 A.M. and 1 P.M. Compute the exact value of this integral.

Solution:

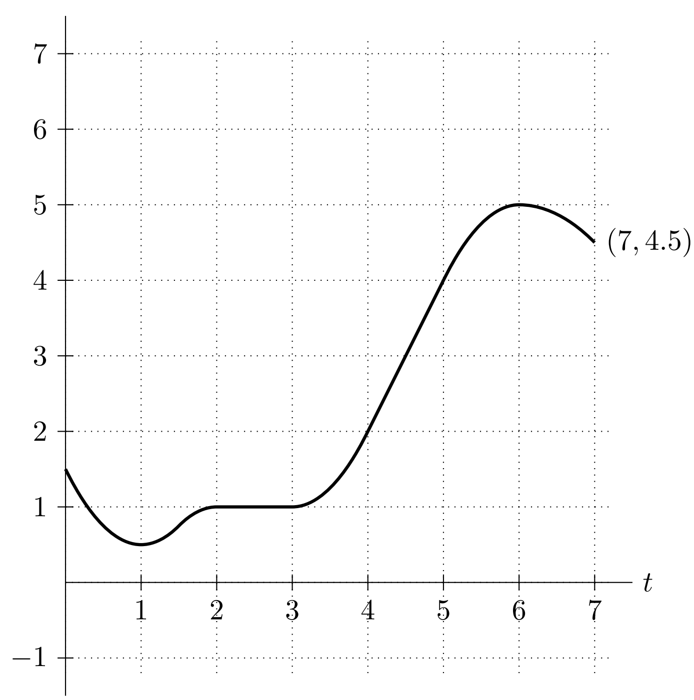
$$\frac{1}{4}\int_1^5 g'(t)\,dt = \frac{1}{4}\left(\frac{7}{2}\right) = \frac{7}{8}$$

b. [2 points] At what time does the tank have the most water in it? At what time does it have the least water?

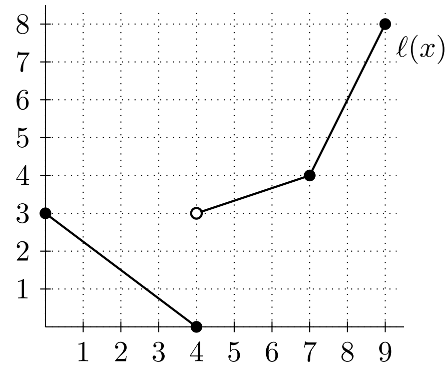
Answer: The tank has the most water in it at **2 P.M.**.

The tank has the least water in it at **9 A.M.**.

c. [6 points] Suppose that $g(3) = 1$. Sketch a detailed graph of $g(t)$ and give both coordinates of the point on the graph at $t = 7$.



4. [10 points] Below is the graph of a piecewise-linear function $\ell(x)$.



For each of the following, circle the correct answer. You do not need to show your work.

a. [2 points] Find $g'(e^2)$, where $g(x) = \ell(\ln x)$.

$-3/4$ $\quad$ $\boxed{-3/(4e^2)}$ $\quad$ $2/e^2$ $\quad$ $-4/3$ $\quad$ 2

b. [2 points] Find $m'(5)$, where $m(x) = \ell(x)\cos(\pi x)$.

$\pi/3$ $\quad$ $-\pi/3$ $\quad$ $1/3$ $\quad$ $\boxed{-1/3}$ $\quad$ 0

c. [2 points] Find $h'(8)$, where $h(x) = \ell(\ell(x))$.

$\boxed{2/3}$ $\quad$ 4 $\quad$ 2 $\quad$ $1/3$ $\quad$ $4/3$

d. [2 points] Find $i'(6)$, where $i(x) = \ell^{-1}(x)$.

$-3/121$ $\quad$ $-9/121$ $\quad$ $1/3$ $\quad$ 2 $\quad$ $\boxed{1/2}$

e. [2 points] Find $j'(2)$, where $j(x) = \dfrac{\ell(x)}{x^2}$.

$-9/4$ $\quad$ $9/4$ $\quad$ $\boxed{-9/16}$ $\quad$ $3/16$ $\quad$ $-3/16$

5. [12 points] Consider the function

$$f(x) = (x - k)e^{-x/k}$$

where k is a positive constant. Note that the derivative of $f(x)$ is

$$f'(x) = e^{-x/k} - \frac{1}{k}(x - k)e^{-x/k}.$$

Your answers to this problem might involve the constant k.

Be sure to show all your work and justify all of your answers.

a. [7 points] Determine the global maximum and minimum values of $f(x)$ on the interval $[0, \infty)$. If $f(x)$ does not have a global maximum or a global minimum on this interval, explain why.

Solution: Begin by finding the critical points of $f(x)$. Since $f(x)$ is differentiable, we just need to find where $f'(x) = 0$.

$$f'(x) = e^{-x/k} - \frac{1}{k}(x - k)e^{-x/k} = e^{-x/k}\left(2 - \frac{x}{k}\right)$$

The factor $e^{-x/k}$ is always positive, and $2 - \frac{x}{k} = 0$ when $x = 2k$. So the only critial point is at $x = 2k$.

Test the critical point, endpoint, and end behavior:

$$f(2k) = ke^{-2}$$
$$f(0) = -k$$
$$\lim_{x\to\infty} f(x) = 0$$

So there is a global maximum value of ke^{-2} at $x = 2k$ and a global minimum value of $-k$ at $x = 0$.

b. [5 points] Find the x-coordinates of all inflection points of $f(x)$ on the domain $[0, \infty)$ or show that $f(x)$ does not have any inflection points on this interval.

Solution: The second derivative of f is

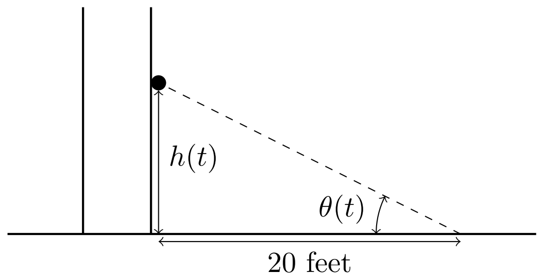
$$f''(x) = -\frac{1}{k}e^{-x/k} - \left(\frac{1}{k}e^{-x/k} - \frac{1}{k^2}(x-k)e^{-x/k}\right) = \frac{1}{k}e^{-x/k}\left(\frac{x}{k} - 3\right)$$

The factor $\frac{x}{k} - 3$ is equal to 0 when $x = 3k$. To show this is an inlection point, we can test $x = 2k$ and $x = 4k$:

$$f''(2k) = -\frac{1}{k}e^{-2} < 0$$
$$f''(4k) = \frac{1}{k}e^{-4} > 0$$

The second derivative changes sign at $x = 3k$ so this is an inflection point.

6. [12 points] Walking through Nichols Arboretum, you see a squirrel running down the trunk of a tree. The trunk of the tree is perfectly straight and makes a right angle with the ground. You stop 20 feet away from the tree and lie down on the ground to watch the squirrel. Suppose $h(t)$ is the distance in feet between the squirrel and the ground, and $\theta(t)$ is the angle in radians between the ground and your line of sight to the squirrel, with t being the amount of time in seconds since you stopped to watch the squirrel.



a. [3 points] Write an equation relating $h(t)$ and $\theta(t)$. (Hint: Use the tangent function.)

Solution:

$$\tan\theta(t) = \frac{h(t)}{20}$$

b. [5 points] If $\theta(t)$ is decreasing at 1/5 of a radian per second when $\theta(t) = \pi/3$, how fast is the squirrel moving at that time?

Solution: Differentiate with respect to t:

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

Plug in $\theta'(t) = -1/5$ and $\theta(t) = \pi/3$:

$$\frac{1}{\cos^2(\pi/3)}(-1/5) = \frac{h'(t)}{20}$$

Solve to get $h'(t) = -16$ so the squirrel is moving at -16 feet per second.

c. [4 points] For the last second before the squirrel reaches the ground, it is moving at a constant speed of 20 feet per second. Suppose $\theta'(t) = -3/4$ at some point during this last second. How high is the squirrel at this time?

Solution: Start with

$$\frac{1}{\cos^2\theta(t)}\theta'(t) = \frac{h'(t)}{20}$$

and plug in $h'(t) = -20$ and $\theta'(t) = -3/4$:

$$\frac{1}{\cos^2\theta(t)}(-3/4) = \frac{-20}{20}$$

This gives us that $\cos^2\theta(t) = 3/4$. Then $\cos\theta(t) = \sqrt{3}/2$ (positive because $0 < \theta < \pi/2$), and $\theta(t) = \arccos(\sqrt{3}/2) = \pi/6$. Finally, we use the equation from part (a) to get

$$h(t) = 20\tan(\pi/6) = 20/\sqrt{3} \approx 11.547$$

so the squirrel is at a height of about 11.547 feet.

7. [10 points] For each of the following statements, circle True if the statement is always true and circle False otherwise. No justification is necessary.

Recall the following definitions:

A function f is *even* if $f(-x) = f(x)$ for all x.

A function f is *odd* if $f(-x) = -f(x)$ for all x.

a. [2 points] If $f(x)$ is an odd function and the tangent line to the graph of $f(x)$ at $x = 2$ is $y = 4(x-2)+7$, then the tangent line to the graph of $f(x)$ at $x = -2$ is $y = -4(x+2)-7$.

True $\boxed{\text{False}}$

b. [2 points] If $g''(x) = 2^x(x-4)(x+5)^2$, then $g(x)$ has inflection points at $x = 4$ and $x = -5$.

True $\boxed{\text{False}}$

c. [2 points] If $h(x)$ is an even function and $\int_{-3}^{8} h(x)\,dx = 17$, then $\int_{-8}^{3} h(x)\,dx = 17$.

$\boxed{\text{True}}$ False

d. [2 points] If $\int_{3}^{7} p(t)\,dt = -5$, then $\int_{-1}^{3} p(t-4)\,dt = -5$.

True $\boxed{\text{False}}$

e. [2 points] If $f(x)$ is a function such that $f''(x)$ is continuous, $f'(3) > 0$, and $f''(3) < 0$, then $f(3+\Delta x) \leq f(3) + f'(3)\Delta x$ for all sufficiently small values of Δx.

$\boxed{\text{True}}$ False

8. [11 points] A basketball player is running sprints in Crisler Center. She begins in the middle of the "M" at the center of the court and runs north and south. Her velocity, in meters per second, for the first 9 seconds is $v(t) = t\sin\left(\frac{\pi}{3}t\right)$, where t is the number of seconds since she started running. She is running north when $v(t)$ is positive and south when $v(t)$ is negative.

a. [3 points] Show that the function

$$f(t) = \frac{9}{\pi^2}\sin\left(\frac{\pi}{3}t\right) - \frac{3}{\pi}t\cos\left(\frac{\pi}{3}t\right)$$

is an antiderivative of $v(t)$.

Solution: To show that $f(t)$ is an antiderivative of $v(t)$, we need to show that $f'(t) = v(t)$.

$$\begin{aligned} f'(t) &= \frac{9}{\pi^2}\cos\left(\frac{\pi}{3}t\right)\frac{\pi}{3} - \frac{3}{\pi}\left(\cos\left(\frac{\pi}{3}t\right) - t\sin\left(\frac{\pi}{3}t\right)\frac{\pi}{3}\right) \\ &= \frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) - \frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + t\sin\left(\frac{\pi}{3}t\right) \\ &= t\sin\left(\frac{\pi}{3}t\right) \end{aligned}$$

b. [3 points] Where on the court is the player after the 9 seconds? Show all your work and give your answer in exact form (no decimal approximations).

Solution: The player starts at the "M" at the center of the court. The player's change in distance is equal to

$$\begin{aligned} \int_0^9 v(t)\,dt &= f(9) - f(0) \\ &= \left(\frac{9}{\pi^2}\sin\left(\frac{\pi}{3}(9)\right) - \frac{3}{\pi}(9)\cos\left(\frac{\pi}{3}(9)\right)\right) - \left(\frac{9}{\pi^2}\sin\left(\frac{\pi}{3}(0)\right) - \frac{3}{\pi}(0)\cos\left(\frac{\pi}{3}(0)\right)\right) \\ &= \frac{27}{\pi} \end{aligned}$$

So the player is $\frac{27}{\pi}$ meters north of the center of the court.

c. [5 points] What is the total distance traveled by the player in the 9 seconds? Show all your work and give your answer in exact form (no decimal approximations).

Solution: Note that $v(t)$ is positive for $0 < t < 3$, negative for $3 < t < 6$, and positive for $6 < t < 9$. Then the total distance traveled by the player is

$$\begin{aligned} \int_0^9 |v(t)|\,dt &= \int_0^3 v(t)\,dt - \int_3^6 v(t)\,dt + \int_6^9 v(t)\,dt \\ &= (f(3) - f(0)) - (f(6) - f(3)) + (f(9) - f(6)) \\ &= \frac{9}{\pi} - \left(-\frac{27}{\pi}\right) + \frac{45}{\pi} \\ &= \frac{81}{\pi} \end{aligned}$$

So the player traveled a distance of $\frac{81}{\pi}$ meters.

9. [10 points] A team of engineers at the university has built a submarine to explore Lake Michigan. The engineers need to keep track of the temperature of the water outside the submarine in order to correctly regulate the engine temperature. The team is working with the following two functions:

- $f(t)$ is the depth of the submarine (in meters) t seconds after the submarine begins its descent. Assume $f(t)$ is invertible on its domain.
- $g(d)$ be the temperature (in degrees Celsius) at a depth of d meters below the surface of Lake Michigan. Assume $g(d)$ is invertible on its domain.

Circle the correct answer to each of the following questions. There is exactly one correct answer for each question.

a. [2 points] Which expression is equal to the time in seconds at which the temperature of the water outside the submarine is 40 degrees?

$f^{-1}(40)$ $\quad$ $g^{-1}(40)$ $\quad$ $\boxed{f^{-1}(g^{-1}(40))}$ $\quad$ $g^{-1}(f^{-1}(40))$

b. [2 points] Three minutes after it begins its descent, the submarine is at a depth of 45 meters. In the next second, the submarine descends approximately 2 meters. Which of the following equations is most likely to be true?

$\boxed{f'(180) = 2}$ $\quad$ $f'(180) = 47$ $\quad$ $f'(45) = 2$ $\quad$ $f'(45) = 47$

c. [2 points] When the submarine is 50 meters below the surface, the temperature of the water outside the submarine will decrease by about 0.2 degrees in the next second. Which of the following equations is most likely to be true?

$\boxed{g'(50)f'(f^{-1}(50)) = -0.2}$ $\quad$ $g'(50) = -0.2$ $\quad$ $g'(f^{-1}(50)) = -0.2$ $\quad$ $g'(g(50)) = -0.2$

d. [2 points] Which expression is equal to the average temperature outside the submarine during the first 10 seconds of the descent?

$\frac{1}{10}\int_0^{10} g(x)\,dx$ $\quad$ $\boxed{\frac{1}{10}\int_0^{10} g(f(x))\,dx}$ $\quad$ $\frac{1}{10}\int_0^{10} g'(x)\,dx$ $\quad$ $\frac{1}{10}\int_0^{10} g'(f(x))\,dx$

e. [2 points] Which expression is equal to the change in temperature during the second minute of the submarine's descent?

$\int_{60}^{120} g(f(x))\,dx$ $\quad$ $\int_{60}^{120} g'(x)\,dx$ $\quad$ $\int_{60}^{120} g'(f(x))\,dx$ $\quad$ $\boxed{\int_{60}^{120} g'(f(x))f'(x)\,dx}$