Math 115 — First Midterm

October 13, 2015

EXAM SOLUTIONS

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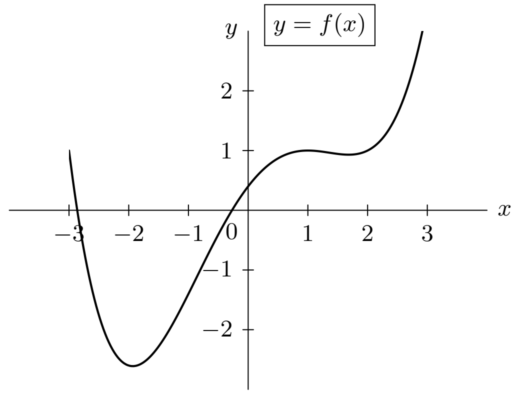
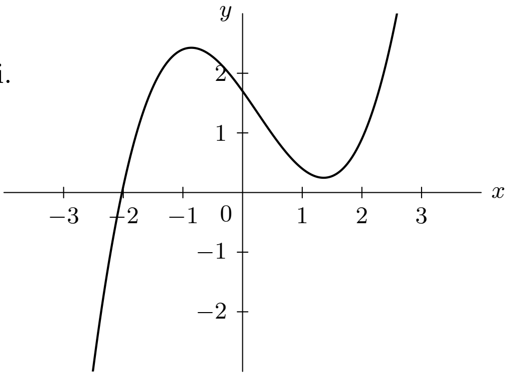
1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your initials (not name) on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. The use of any networked device while working on this exam is not permitted.
7. You may use any calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones and smartwatches.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	5	
2	12	
3	9	
4	13	
5	8	
6	6	
7	12	
8	12	
9	13	
10	10	
Total	100	

1. [5 points] Below is the graph of a function $f(x)$.



There are six graphs shown below. Circle the <u>one</u> graph that could be the graph of the derivative $f'(x)$.

i.


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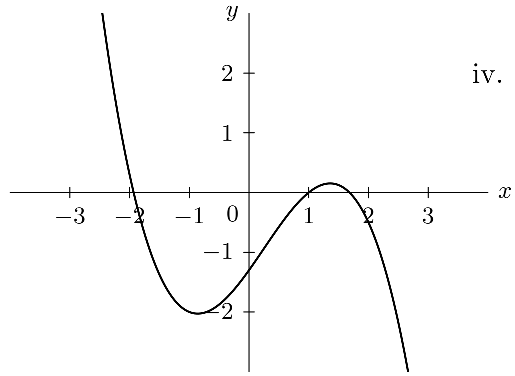

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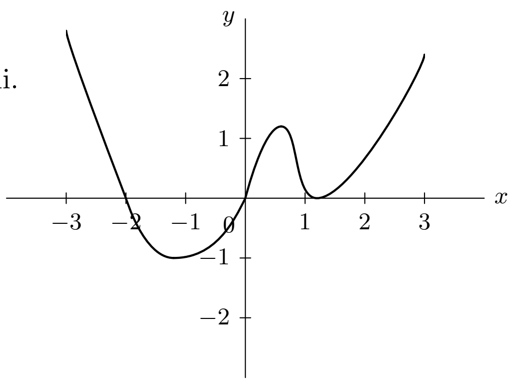

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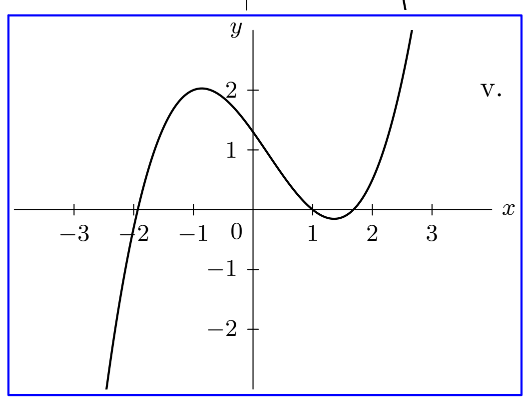

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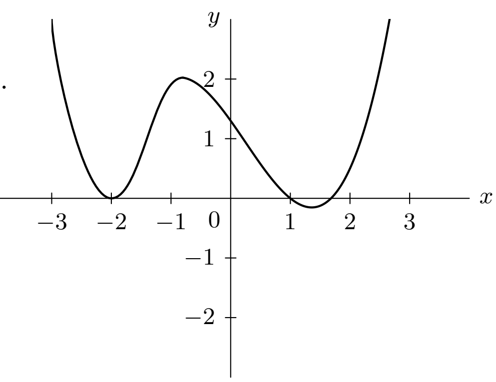

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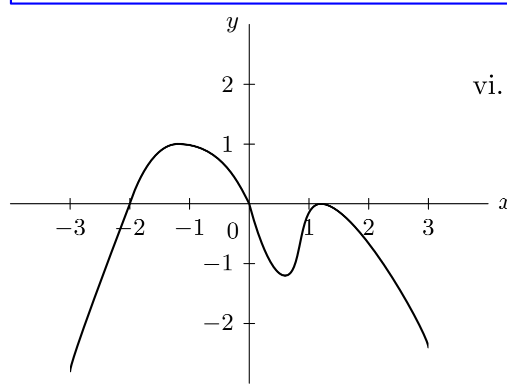


2. [12 points] Angelica Neiring and Simona Koloji decide to enjoy the fall weather by racing each other from the brass block "M" in the center of the Diag along a 2.5 kilometer (2500 meter) route to the Huron River inside the Arb. Let $A(t)$ (respectively $S(t)$) be Angelica's (respectively Simona's) distance along the route (in meters) t seconds after they start racing. Angelica and Simona are both wearing GPS watches that record data about their race. The table of values for the functions A and S below shows some of the resulting data, rounded to the nearest meter. Note that the data is not always recorded at regular intervals.

t	0	30	60	66	72	105	114	120	135	168	180	198	300
$A(t)$	0	55	119	137	156	226	249	265	302	384	415	463	737
$S(t)$	0	57	120	137	156	225	248	264	303	389	422	473	768

Use the data above to answer the questions below. Remember to show your work.

a. [2 points] Estimate Angelica's instantaneous velocity 3 minutes into the race. *Include units.*

Solution: We estimate using average velocity based on nearby measurements:

Avg velocity between 168 and 180 seconds: $\dfrac{A(180)-A(168)}{180-168} = \dfrac{415-384}{12} = \dfrac{31}{12} \approx 2.58$ m/sec

Avg velocity between 180 and 198 seconds: $\dfrac{A(198)-A(180)}{198-180} = \dfrac{463-415}{18} = \dfrac{48}{18} \approx 2.67$ m/sec

So we estimate that Angelica's instantaneous velocity 3 minutes into the race was about 2.6 meters per second. (Any estimate between the average velocities computed above would be reasonable. We rounded to two significant digits.)

Answer: **About 2.6 meters per second**

b. [2 points] Estimate $S'(120)$.

Solution: We estimate the derivative using nearby measurements:

Average rate of change of $S(t)$ for $114 \le t \le 120$: $\dfrac{S(120)-S(114)}{120-114} = \dfrac{264-248}{6} = \dfrac{16}{6} \approx 2.67$

Average rate of change of $S(t)$ for $120 \le t \le 135$: $\dfrac{S(135)-S(120)}{135-120} = \dfrac{303-264}{15} = \dfrac{39}{15} \approx 2.6$

So we estimate that $S'(120) \approx 2.6$. (Any estimate between the average rates of change computed above would be reasonable. We rounded to two significant digits here.)

Answer: $S'(120) \approx 2.6$

c. [2 points] In the context of this problem, what are the units on the quantity $(A^{-1})'(150)$?

Answer: **seconds per meter**

For questions **d.** and **e.** below, circle the one best answer or circle CANNOT BE DETERMINED if there is not enough information to definitely determine the answer. You do not need to show your work or provide justification for your answers for these questions.

d. [1 point] Who was ahead 5 minutes into the race?

Angelica $\quad$ $\boxed{\text{Simona}}$ $\quad$ CANNOT BE DETERMINED

e. [1 point] Who was running faster exactly one minute into the race?

Angelica $\quad$ Simona $\quad$ $\boxed{\text{CANNOT BE DETERMINED}}$

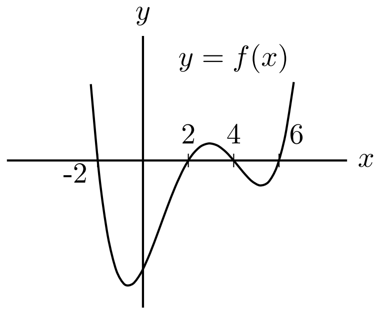
f. [4 points] In describing the race later, Simona says that her average velocity during the entire race was 2.8 meters per second while Angelica says that after the first 5 minutes, her average velocity for the rest of the race was 3.1 meters per second.
Assuming their statements and the table of values above are accurate, who won the race? Or is there not enough information to decide? Explain your reasoning.

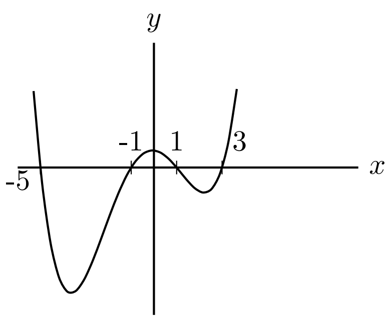
Answer: (Circle one.) $\boxed{\text{Angelica}}$ Simona Not enough information

Explanation:

Solution: With an average velocity of 2.8 meters per second for the whole race of 2500 meters, it took Simona $\dfrac{2500}{2.8} \approx 892.9$ seconds to complete the 2500 meter race. On the other hand, it took Angelica 300 seconds to run the first 737 meters and, with an average velocity of 3.1 meters per second for the rest of the race $(2500 - 737$ meters), it took her an additional $\frac{2500-737}{3.1} \approx 568.7$ seconds to finish the race. So Angelica's total time for the race was about $300 + 568.7 = 867.7$ seconds, which is less than Simona's time of about 892.9 seconds. Hence Angelica won the race.

3. [9 points] A portion of the graph of a function f is shown below, along with three graphs obtained from f by one or more transformations. Below each of the three graphs is a list of possible formulas for that graph. Find the <u>one</u> correct formula for each graph, and write the corresponding letter in the answer blank provided.
Note that the zeros of each graph are labeled and that the scales on both the vertical and horizontal axes are the same for all the graphs shown.





Answer: **H**

A. $f(x-1)$

B. $f(x-2)$

C. $f(x-3)$

D. $f(-(x-2))$

E. $f(-(x-3))$

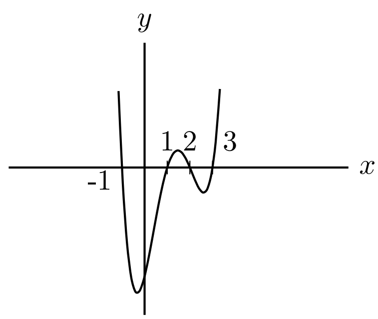
F. $f(x+1)$

G. $f(x+2)$

H. $\boxed{f(x+3)}$

I. $-f(x+1)$

J. $-f(x+3)$



Answer: **E**

A. $f(x-1)$

B. $f(x+1)$

C. $f(\frac{1}{3}x)$

D. $f(\frac{1}{2}x)$

E. $\boxed{f(2x)}$

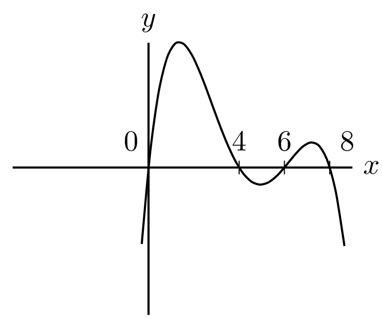
F. $f(3x)$

G. $f(2(x+1))$

H. $f(\frac{1}{2}(x-1))$

I. $f(2(x-1))$

J. $f(\frac{1}{2}(x+1))$



Answer: **E**

A. $f(x+2)$

B. $f(x-2)$

C. $f(-(x-2))$

D. $-f(x+1)$

E. $\boxed{-f(x-2)}$

F. $-f(x+2)$

G. $-f(x)$

H. $f(-(x+2))$

I. $-f(-x)$

J. $f(-(x-4))$

4. [13 points] Algernon Brayik is making scones. He knows that the height of a scone is a function of how much baking soda it contains. Let $h(B)$ be the height in millimeters of a scone that contains B grams of baking soda. Assume that the function h is increasing and invertible, and that h and h^{-1} are both differentiable.

a. [2 points] Algie looks in his baking soda container and finds that there are exactly 46 grams of baking soda remaining. Suppose he uses all of this baking soda to make 8 scones, and that the baking soda is equally distributed among all 8 of the scones. Write a mathematical expression involving h or h^{-1} for the height (in millimeters) of each resulting scone.

Answer: $h\left(\frac{46}{8}\right)$

b. [5 points] Below is the first part of a sentence that will give a practical interpretation of the equation $h'(6) = 15$ in the context of this problem. Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include units in your answer.

If Algie decreases the amount of baking soda per scone from 6 grams to 5.8 grams, then...

Solution: If Algie decreases the amount of baking soda per scone from 6 grams to 5.8 grams, then the height of each scone will decrease by approximately 3 millimeters.

c. [3 points] Algie makes a batch of scones, with each scone containing k grams of baking soda (for some constant k). When the scones come out of the oven, he decides they are each 10 millimeters shorter than he would like. Write a mathematical expression involving k, h, and h^{-1} for the number of grams of baking soda per scone he should use to get scones of the desired height.

Answer: $h^{-1}(h(k) + 10)$

d. [3 points] Algie does some calculations and determines that $\dfrac{60}{h^{-1}(30)} = 40$.
Based on this information, which of the following statements must be true?
Circle <u>all</u> of the statements that must be true or circle NONE OF THESE.

A. If Algie makes 40 scones, each with 30 grams of baking soda, then the scones will rise to a height of 60 millimeters.

B. If Algie wants to make 40 scones, then he must use 60 grams of baking soda.

C. If Algie wants to make scones of height 30 millimeters and he has 60 grams of baking soda, then the maximum number of scones he can make is 40.

D. A scone containing 1.5 grams of baking soda rises to a height of 30 millimeters.

E. A scone containing 30 grams of baking soda rises to a height of 1.5 millimeters.

F. NONE OF THESE

5. [8 points] *Remember to show your work carefully throughout this problem.*
Algie and Cal go on a picnic, arriving at 12:00 noon.

a. [5 points] Five minutes after they arrive, they notice that 5 ants have joined their picnic. More ants soon appear, and after careful study, they determine that the number of ants appears to be increasing by 20% every minute. Find a formula for a function $A(t)$ modeling the number of ants present at the picnic t minutes past noon for $t \geq 5$.

Solution: Since this is an exponential function, there are constants c and b such that $A(t) = cb^t$. We can see immediately that $b = 1.2$. We can then use the fact that we know that $A(5) = 5$ to find c: $c(1.2)^5 = 5$, so $c = 5/(1.2)^5$, which is approximately 2.01. Alternatively, we can use a horizontal shift to say that this is $5(1.2)^{t-5}$.

Answer: $A(t) =$ $5(1.2)^{(t-5)} = \frac{5}{1.2^5}(1.2)^t$

b. [3 points] Algie and Cal notice that their food is, unfortunately, also attracting flies. The number of flies at their picnic t minutes after noon can be modeled by the function $g(t) = 1.8(1.25)^t$. Algie and Cal decide they will end their picnic when there are at least 1000 flies. How long will their picnic last? *Include units.*

Solution: We wish to find t such that $1.8(1.25)^t = 1000$. Then

$$\begin{aligned} 1.8(1.25)^t &= 1000 \\ \ln(1.8(1.25)^t) &= \ln(1000) \\ \ln(1.8) + t\ln(1.25) &= \ln(1000) \\ t\ln(1.25) &= \ln(1000) - \ln(1.8) \\ t &= \ln(1000/1.8)/\ln(1.25) \approx 28.3. \end{aligned}$$

So they end their picnic about 28.3 minutes after noon (when it started).

Answer: **About 28.3 minutes**

6. [6 points] Consider the function

$$R(w) = 2 + (\ln(w))^{\cos(w)}.$$

Use the limit definition of the derivative to write an explicit expression for $R'(\pi)$.
Your answer should not involve the letter R. Do not attempt to evaluate or simplify the limit.
Please write your final answer in the answer box provided below.

Answer: $R'(\pi) =$ $\boxed{\lim_{h\to 0} \frac{\left(2 + (\ln(\pi+h))^{\cos(\pi+h)}\right) - \left(2 + (\ln(\pi))^{\cos(\pi)}\right)}{h}}$

7. [12 points] Phillip Asafy and Genevieve Omicks both enjoy hot chocolate when it's cool outside. They made a few measurements, and these appear in the table below.

P (respectively G) is Phil's (respectively Gen's) consumption of hot chocolate (in quarts, measured to the nearest tenth of a quart) in a month when the average daily high temperature is H (in degrees Celsius, measured to the nearest degree).

H(°C)	P (quarts)	G (quarts)
3	16.1	13.3
7	12.8	11.6
15	8.0	6.5

a. [8 points] Based on this data, could either student's monthly hot chocolate consumption be reasonably modeled as a linear function of average daily high temperature? An exponential function? Neither? Carefully justify your answer in the space below.
(Hint: *At least* one of these can be modeled by a linear or an exponential function!)

Solution: First consider Phil's hot chocolate consumption. Suppose $P = p(H)$. To check whether $p(H)$ could be modeled by a linear function, we compute the average rate of change of p over the intervals $[3, 7]$ and $[7, 15]$. We have

$$\frac{p(7) - p(3)}{7 - 3} = \frac{12.8 - 16.1}{4} = -0.825 \quad \text{and} \quad \frac{p(15) - p(7)}{15 - 7} = \frac{8.0 - 12.8}{8} = -0.6.$$

Since these two average rates of change are quite different, Phil's hot chocolate consumption is not reasonably modeled by a linear function. To check whether $p(H)$ could be modeled by an exponential function, we compute the percent rate of change of $p(H)$ over the intervals $[3, 7]$ and $[7, 15]$. We have

$$\left(\frac{p(7)}{p(3)}\right)^{\frac{1}{7-3}} = \left(\frac{12.8}{16.1}\right)^{\frac{1}{4}} \approx 0.9443 \quad \text{and} \quad \left(\frac{p(15)}{p(7)}\right)^{\frac{1}{15-7}} = \left(\frac{8.0}{12.8}\right)^{\frac{1}{8}} \approx 0.9429.$$

The difference between these percent rates of change is less than 0.2%, so based on this data, $p(H)$ can be reasonably modeled by an exponential function. In particular, we can check that we obtain the data in the table for P using, for example, $19.2(0.9435)^H$
Now consider Gen's hot chocolate consumption. Suppose $G = g(H)$. From the calculations

$$\frac{g(7) - g(3)}{7 - 3} = \frac{11.6 - 13.3}{4} = -0.425 \quad \text{and} \quad \frac{g(15) - g(7)}{15 - 7} = \frac{6.5 - 11.6}{8} = -0.6375$$

we conclude that Gen's hot chocolate consumption is not reasonably modeled by a linear function. From the calculations

$$\left(\frac{g(7)}{g(3)}\right)^{\frac{1}{7-3}} = \left(\frac{11.6}{13.3}\right)^{\frac{1}{4}} \approx 0.9664 \quad \text{and} \quad \left(\frac{g(15)}{g(7)}\right)^{\frac{1}{15-7}} = \left(\frac{6.5}{11.6}\right)^{\frac{1}{8}} \approx 0.9302$$

we conclude that Gen's hot chocolate consumption can't be reasonably modeled by an exponential function.
(Note that for the exponential cases we could instead compare, for example, $\left(\frac{p(7)}{p(3)}\right)^2$ with $\frac{p(15)}{p(7)}$.)

Answers: Circle one choice for each student.

Phil's consumption P: linear $\boxed{\text{exponential}}$ neither linear nor exponential

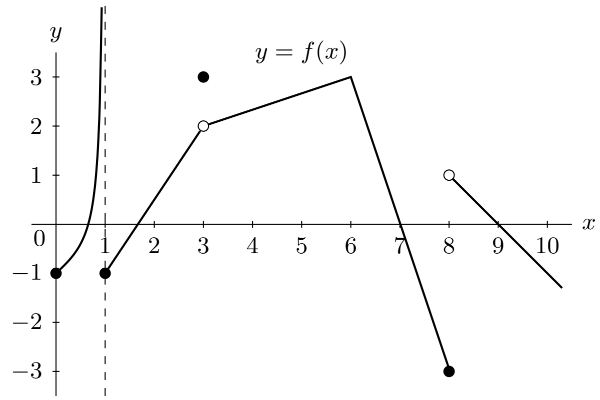
Gen's consumption G: linear exponential $\boxed{\text{neither linear nor exponential}}$

b. [4 points] For this investigation, their friend Maddy measures temperature in degrees Fahrenheit, and she measures her hot chocolate consumption in cups. She finds a function $M(f)$ which is the number of cups of hot chocolate she consumes in a month when the average daily high temperature is f degrees Fahrenheit. If $Q(H)$ is the number of *quarts* of hot chocolate Maddy consumes when the average monthly temperature is H degrees *Celsius*, write a formula for $Q(H)$ in terms of M and H.
Recall that there are 4 cups in a quart and that the conversion from Fahrenheit to Celsius is given by $y = \frac{5}{9}(x - 32)$ *(where* $y^\circ C$ *and* $x^\circ F$ *describe the same temperature).*

Solution: H degrees Celsius is the same as $\dfrac{9}{5}H+32$ degrees Fahrenheit, and $M\left(\dfrac{9}{5}H + 32\right)$ gives Maddy's hot chocolate consumption in cups. We divide this quantity by 4 to convert from cups to quarts.

Answer: $Q(H) = \dfrac{M\left(\frac{9}{5}H + 32\right)}{4}$

8. [12 points] A portion of the graph of a function f is shown below.



a. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x \to c} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ 1, 8

b. [2 points] Give all values c in the interval $0 < c < 10$ for which $\lim_{x \to c^+} f(x)$ does not exist. If there are none, write NONE.

Answer: $c =$ **NONE**

c. [2 points] Give all values c in the interval $0 < c < 10$ for which $f(x)$ is not continuous at c. If there are none, write NONE.

Answer: $c =$ 1, 3, 8

d. [6 points] With f as shown in the graph above, define a function g by the formula

$$g(x) = \begin{cases} \dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} & \text{if } x \le 0 \\ f(x) & \text{if } 0 < x < 10 \end{cases}$$

where A and B are nonzero constants.
Find values of A and B so that both of the following conditions hold.

- $g(x)$ is continuous at $x = 0$.
- $\lim_{x \to -\infty} g(x) = \frac{1}{2}$.

If no such values exist, write NONE in the answer blanks.
Be sure to show your work or explain your reasoning.

Solution: To satisfy the first condition, we first compute $g(0)$ by plugging in $x = 0$ to the rational function $\dfrac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5}$ to find $\lim_{x \to 0-} g(x) = g(0) = \dfrac{B}{12}$. In order for $g(x)$ to be continuous at $x = 0$, we must also have $\lim_{x \to 0^+} g(x) = \dfrac{B}{12}$.
Now $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} f(x) = -1$ (from the graph), so $\frac{B}{12} = -1$, and $B = -12$.
To satisfy the second condition, we compute that

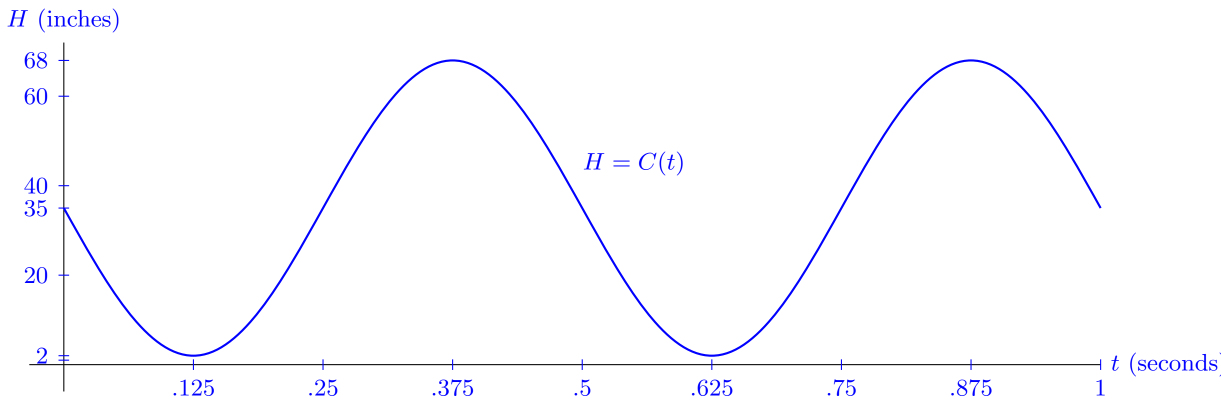
$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{B + 2x^2 + 3x^3 + Ax^5}{12 + 6x^3 + 4x^5} = \frac{A}{4}.$$

In order for this limit to equal $\frac{1}{2}$, we must have $A/4 = 1/2$, so $A = 2$.

Answer: $A =$ 2 and $B =$ -12

9. [13 points] Cal is jumping a rope being swung by Gen and Algie while Maddy runs a stopwatch. There is a piece of tape around the middle of the rope. When the rope is at its lowest, the piece of tape is 2 inches above the ground, and when the rope is at its highest, the piece of tape is 68 inches above the ground. The rope makes two complete revolutions every second. When Maddy starts her stopwatch, the piece of tape is halfway between its highest and lowest points and moving downward. The height H (in inches above the ground) of the piece of tape can be modeled by a sinusoidal function $C(t)$, where t is the number of seconds displayed on Maddy's stopwatch.

a. [4 points] On the axes provided below, sketch a well-labeled graph of two periods of $C(t)$ beginning at $t = 0$.
Pay attention to both the shape of your graph and the location of important points.



b. [4 points] Find a formula for $C(t)$.

Answer: $C(t) =$ $35 - 33\sin(4\pi t)$

c. [5 points] Now Gen takes a turn at jumping while Cal and Algie swing the rope. Maddy resets the stopwatch and starts it over again. Let $G(w)$ be the height (in inches above the ground) of the piece of tape when Maddy's stopwatch says w seconds. A formula for $G(w)$ is

$$G(w) = 41 + 38\cos(2\pi w).$$

Maddy is 60 inches tall. For how long (in seconds) during each revolution of the rope is the piece of tape higher than the top of Maddy's head? (Assume Maddy is standing straight while watching the stopwatch.) *Remember to show your work.*

Solution: We are looking for when $41 + 38(\cos(2\pi w) > 60$.
First we find one time when $41 + 38(\cos(2\pi w) = 60$.

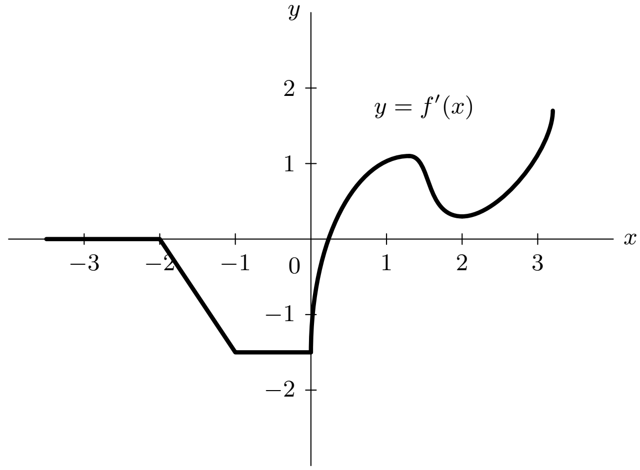
$$\begin{aligned} 38\cos(2\pi w) &= 19 \\ \cos(2\pi w) &= 1/2 \\ 2\pi w &= \arccos(1/2) \text{ (is one solution)} \\ 2\pi w &= \pi/3 \\ w &= 1/6 \end{aligned}$$

There are many ways to find the answer from here. One way is to note that since we are looking at a cosine function that has not been shifted horizontally, for $0 < w < 1/6$, the rope is above her head, so it's there for the first sixth of a second. By symmetry around the peak, the rope reached that height $1/6$ second before the timer started. Thus the rope is above her head for $2 \cdot 1/6 = 1/3$ second during each revolution.

Answer: $\frac{1}{3}$ **second**

10. [10 points] Below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.

Note that $f'(x)$ is zero for $x \le -2$, linear for $-2 < x < -1$, and constant for $-1 < x < 0$.



For each of the following, circle <u>all</u> of the listed intervals for which the given statement is true over the **entire** interval. If there are no such intervals, circle NONE.
You do not need to explain your reasoning.

a. [2 points] $f'(x)$ is increasing.

$-2 < x < -1$ $\quad \boxed{0 < x < 1}$ $\quad 1 < x < 2$ $\quad \boxed{2 < x < 3}$ $\quad$ NONE

b. [2 points] $f'(x)$ is concave up.

$0 < x < 1$ $\quad 1 < x < 2$ $\quad \boxed{2 < x < 3}$ $\quad$ NONE

c. [2 points] $f(x)$ is increasing.

$-2 < x < -1$ $\quad -1 < x < 0$ $\quad 0 < x < 1$ $\quad \boxed{1 < x < 2}$ $\quad \boxed{2 < x < 3}$ $\quad$ NONE

d. [2 points] $f(x)$ is linear but not constant.

$-3 < x < -2$ $\quad -2 < x < -1$ $\quad \boxed{-1 < x < 0}$ $\quad 0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE

e. [2 points] $f(x)$ is constant.

$-3 < \boxed{x < -2}$ $\quad -2 < x < -1$ $\quad -1 < x < 0$ $\quad 0 < x < 1$ $\quad 1 < x < 2$ $\quad 2 < x < 3$ $\quad$ NONE