Math 115 — First Midterm — October 10, 2016

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	11	
3	10	
4	10	
5	10	

Problem	Points	Score
6	12	
7	10	
8	7	
9	12	
10	8	
Total	12	

1. [10 points] Laquita decides to visit an amusement park during Fall Break and rides several roller coasters, including the Classic Amazing Looping Coaster and the Ultra Mountain.
Let $R(t)$ be the distance, in feet, that the CAL Coaster has moved along the track t seconds after the ride begins. The ride lasts a total of 60 seconds. Several values of $R(t)$ are shown in the following table.

t	0	10	25	30	40	45	55	60
$R(t)$	0	496	1103	1327	1817	2136	2718	3141

For parts **a.**– **c.**, *remember to show your work and reasoning clearly.*

a. [2 points] Find the average velocity of the CAL Coaster during the last 15 seconds of the ride, i.e. for $45 \le t \le 60$. *Include units.*

Solution: $\dfrac{R(60)-R(45)}{60-45} = \dfrac{3141-2136}{60-45} = \dfrac{1005}{15} = 67.$

Answer: 67 **ft/sec**

b. [2 points] Estimate the instantaneous velocity of the CAL Coaster 30 seconds after the ride begins. *Include units.*

Solution: We estimate using average velocity based on nearby measurements.
Average velocity for $25 \le t \le 30$: $\dfrac{R(30)-R(25)}{30-25} = \dfrac{1327-1103}{5} = \dfrac{224}{5} = 44.8$
(Note that other answers, such as those incorporating $(40, 1817)$, were accepted if work and appropriate units were shown.)

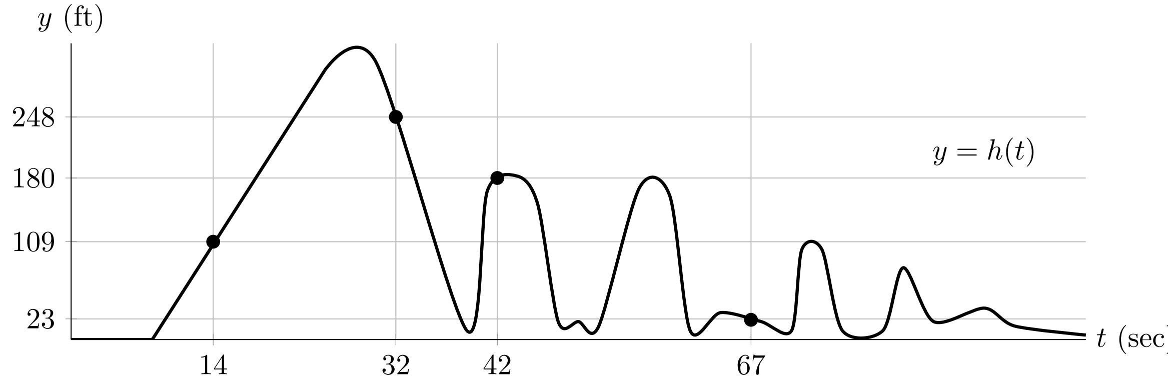
Answer: **About** 44.8 **ft/sec**

c. [2 points] Estimate $R'(55)$.

Solution: We estimate the derivative using the average rate of change of $R(t)$ for $55 \le t \le 60$: $\dfrac{R(60)-R(55)}{60-55} = \dfrac{3141-2718}{5} = 84.6$
(Again, answers incorporating $(45, 2136)$ were accepted with appropriate work.)

Answer: $R'(55) \approx$ 84.6

d. [4 points] Let $h(t)$ be Laquita's height, in feet, above the ground, t seconds after her ride on the Ultra Mountain begins. A graph of $h(t)$ is shown below.



Let the quantities I–V be defined as follows:

I. The number 0.

II. Laquita's instantaneous vertical velocity, in ft/sec, at $t = 14$.

III. $h'(32)$

IV. Laquita's average vertical velocity, in ft/sec, between $t = 14$ and $t = 42$.

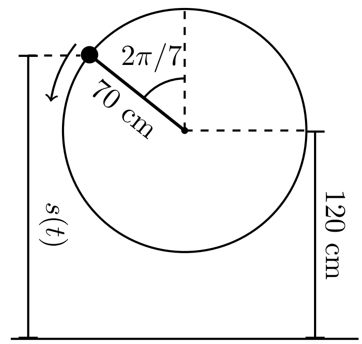
V. Laquita's instantaneous vertical velocity, in ft/sec, at $t = 67$.

Rank the quantities in order from least to greatest by filling in the blanks below with the options I–V. You do not need to show your work.

III < **V** < **I (0)** < **IV** < **II**

2. [11 points] Note that the situations in parts **a.** and **b.** are not related.

a. [6 points] In her latest trick, Dorraine swings a glow toy in a vertical circle (i.e., perpendicular to the ground). The glow toy starts to glow when it swings $2\pi/7$ radians past the top of the circle. The glow toy is attached to one end of a 70 cm rope, and Dorraine holds the other end at a constant height of 120 cm above the ground. The glow toy rotates at a constant rate, making 13 revolutions in 5 seconds. Let $s(t)$ be the height in cm above the ground of the glow toy t seconds after the glow toy starts to glow.



Find a formula for $s(t)$.

Note that there are many possible solutions. Two are given below.

Answer: $s(t) =$ $120 + 70\cos\left(\frac{26\pi}{5}t + \frac{2\pi}{7}\right)$ or $120 + 70\sin\left(\frac{26\pi}{5}t + \frac{2\pi}{7} + \frac{\pi}{2}\right)$

b. [5 points] Later, Dorraine swings a handmade toy. The height in cm above the ground of the handmade toy t seconds after she begins swinging it is given by

$$h(t) = 130 + 50\cos\left(\frac{10\pi}{7}t + \frac{\pi}{5}\right).$$

Compute the two smallest positive values of t at which the handmade toy was 160 cm above the ground. Clearly show each step of your work. Give your answers in exact form.

Solution: One solution is found using arccos:

$$130 + 50\cos\left(\frac{10\pi}{7}t + \frac{\pi}{5}\right) = 160$$

$$\cos\left(\frac{10\pi}{7}t + \frac{\pi}{5}\right) = 0.6$$

One solution is given by $\frac{10\pi}{7}t + \frac{\pi}{5} = \arccos(0.6)$

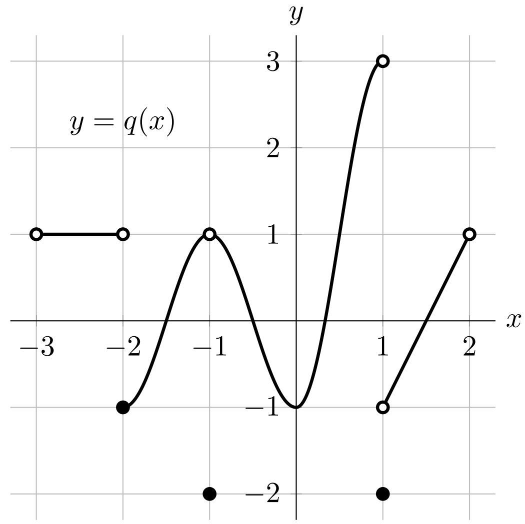
$$t = \frac{7}{10\pi}\left(\arccos(0.6) - \frac{\pi}{5}\right)$$

Verify that this is in fact the smallest positive solution.
The next positive solution of t (found using the unit circle or symmetry) is

$$t = \frac{7}{10\pi}\left(2\pi - \arccos(0.6) - \frac{\pi}{5}\right).$$

Answer: $t =$ $\frac{7}{10\pi}\left(\arccos(0.6) - \frac{\pi}{5}\right)$ and $\frac{7}{10\pi}\left(2\pi - \arccos(0.6) - \frac{\pi}{5}\right)$

3. [10 points] The entire graph of a function q is shown below. Note that $q(x)$ is linear on the interval $1 < x < 2$.



Throughout this problem, you do not need to explain your reasoning.

For each of parts **a.**– **c.** below, circle <u>all</u> of the listed values satisfying the given statement. If there are no such values, circle NONE.

a. [2 points] For which of the following values of a does $\lim_{t\to a} q(t)$ exist?

$a = -2$ | $\boxed{a = -1}$ | $\boxed{a = 0}$ | $a = 1$ | NONE

b. [2 points] For which of the following values of b is $q(x)$ continuous at $x = b$?

$b = -2$ | $b = -1$ | $\boxed{b = 0}$ | $b = 1$ | NONE

c. [2 points] For which of the following values of c is $\lim_{x\to c^+} q(x) = q(c)$?

$\boxed{c = -2}$ | $c = -1$ | $\boxed{c = 0}$ | $c = 1$ | NONE

For each of parts **d.** and **e.** below, if the limit does not exist (including the case of limits that diverge to ∞ or $-\infty$), write DNE.

d. [2 points] Evaluate the following expression: $\lim_{k\to 0} \dfrac{q(1.21 + k) - q(1.21)}{k}$.

Answer: 2

e. [2 points] Evaluate the following expression: $\lim_{s\to -1} q(q(s))$.

Answer: 3

4. [10 points] Consider the function f defined by $f(x) = \dfrac{(x+1.8)(x+2.1)}{(2x+1.8)(3x-6.9)(x+2.1)}$.

You do not have to show your work/reasoning on this problem. However, any work that you do show may be considered for partial credit.

a. [3 points] What is the domain of f?

Answer: **all real numbers except** -0.9**,** 2.3**, and** -2.1

b. [2 points] Find the equations of all vertical asymptotes of the graph of $y = f(x)$. If there are none, write NONE.

Answer: $x = -0.9$ **and** $x = 2.3$

c. [2 points] Let $g(x) = e^{-0.4x}$.

Find the equations of all horizontal asymptotes of the graph of $y = \dfrac{g(x)}{f(x)}$.

If there are none, write NONE.

Solution: $g(x)$ is a positive exponential decay function and dominates any rational function as $x \to \infty$. In particular, $\lim_{x\to\infty} \dfrac{g(x)}{f(x)} = 0$ and $\lim_{x\to-\infty} \dfrac{g(x)}{f(x)} = \infty$ (DNE), so the only horizontal asymptote of the graph of $y = \dfrac{g(x)}{f(x)}$ is $y = 0$.

Answer: $y = 0$

d. [3 points] Find a formula for a rational function $h(x)$ such that $\lim_{x\to\infty} \dfrac{f(x)}{h(x)} = 8$.

Solution: There are many possible answers. Some examples include:

- $h(x) = \dfrac{1}{8 \cdot 6 \cdot x} = \dfrac{1}{48x}$, and
- $h(x) = \dfrac{1}{8} f(x) = \dfrac{(x+1.8)(x+2.1)}{8(2x+1.8)(3x-6.9)(x+2.1)}$.

Answer: $h(x) = \dfrac{1}{48x}$

5. [10 points] Scientists bore a hole deep into the earth and lower an instrument to record the temperature. As the instrument goes deeper, the temperature it records increases. Let $T = g(w)$ be the temperature, in degrees Celsius, the instrument records when it is w hectometers below the surface of the earth. (Recall that 1 hectometer is 100 meters.) Assume that the function g is invertible and that the functions g and g^{-1} are continuous and differentiable.

a. [3 points] Using a complete sentence, give a practical interpretation of the equation $g^{-1}(68) = 49$ in the context of this problem. Be sure to include units.

Solution: When the instrument records a temperature of 68°C, the instrument is 4.9 kilometers below the surface of the earth.

b. [4 points] Below is the first part of a sentence that will give a practical interpretation of the equation $g'(13) = 0.6$ in the context of this problem. Complete the sentence so that the practical interpretation can be understood by someone who knows no calculus. Be sure to include units in your answer.

When the instrument is lowered from 1300 meters to 1320 meters below the surface of the earth, the temperature it records . . .

Solution: increases by approximately 0.12 Celsius degrees.

c. [3 points] Circle the <u>one</u> statement below that is best supported by the equation

$$(g^{-1})'(56) = 0.4.$$

i. The temperature recorded by the instrument is 56°C when it is about 0.4 hectometers below the surface of the earth.

ii. The temperature recorded by the instrument increases from 56°C to 56.4°C when the instrument is lowered approximately one more hectometer.

iii. When the instrument is lowered from 55.9 hectometers to 56 hectometers below the surface of the earth, it detects an increase in temperature of about 0.04 Celsius degrees.

iv. The temperature recorded by the instrument increases from 56°C and 57°C when the instrument is lowered about $\frac{1}{0.4}$ (= 2.5) hectometers further.

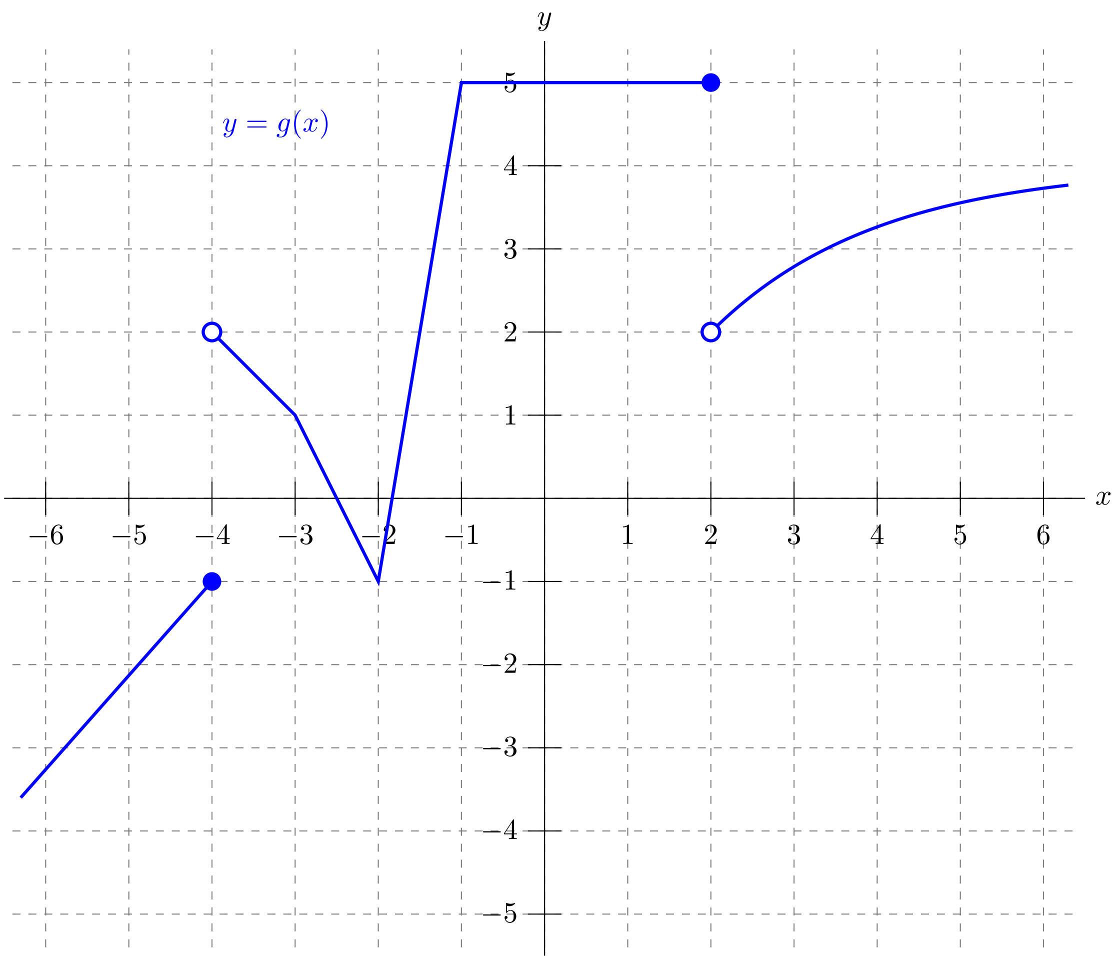
v. As the temperature recorded by the instrument increases from 55.9°C to 56°C, the instrument is lowered about 4 meters further beneath the surface of the earth.

vi. When the instrument is 56 hectometers below the surface of the earth, the recorded temperature is increasing at a rate of 0.04 Celsius degrees per meter.

6. [12 points] On the axes provided below, sketch the graph of a single function $y = g(x)$ satisfying all of the following:

- $g(x)$ is defined for all x in the interval $-6 < x < 6$.
- For all x in the interval $-6 < x < -4$, the function $g(x)$ is continuous at x and $g'(x) > 0$.
- $g(-4) = -1$.
- $\lim_{x \to -4^+} g(x) = 2$.
- $g(-3) = 1$.
- $g(-2) = -1$.
- The function $g(x)$ is continuous on the interval $[-3, -1]$.
- The average rate of change of $g(x)$ between $x = -3$ and $x = -1$ is 2.
- $g'(1) = 0$.
- $g(x)$ is <u>not</u> continuous at $x = 2$.
- The function $g(x)$ is continuous on the interval $3 < x < 6$.
- The slope of the tangent line to the graph of $y = g(x)$ at $x = 3$ is positive.
- $g(x)$ is increasing and concave down on the interval $4 < x < 6$.

Make sure that your graph is large and unambiguous. Note that many solutions are possible.



7. [10 points] Let $N(u) = \begin{cases} e + 3^{u^2+k} & \text{if } u < 1 \\ 5e\ln(e+u-1) & \text{if } u \geq 1, \end{cases}$ where k is a constant.

a. [6 points] Use the limit definition of the derivative to write an explicit expression for $N'(-2)$. *Your answer should not involve the letter N. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Answer: $N'(-2) = \lim_{h\to 0} \frac{e + 3^{(-2+h)^2+k} - (e + 3^{(-2)^2+k})}{h}$

b. [4 points] Find all values of k so that $N(u)$ is continuous at $u = 1$.
Show your work carefully, and leave your answer(s) in exact form.

Solution: To be continuous at $u = 1$, the left and right limits of $N(u)$ at 1 must both be equal to $N(1)$. In particular, k must satisfy the following equation:

$$e + 3^{1^2+k} = 5e\ln(e+1-1).$$

Solving for k, we find:

$$\begin{aligned} e + 3^{1+k} &= 5e \cdot 1 \\ 3^{1+k} &= 4e \\ \ln(3^{1+k}) &= \ln(4e) \\ (k+1)\cdot\ln(3) &= \ln(4) + \ln(e) \\ k+1 &= \frac{\ln(4)+1}{\ln(3)} \\ k &= \frac{\ln(4)+1}{\ln(3)} - 1 \end{aligned}$$

Answer: $k = \frac{\ln(4)+1}{\ln(3)} - 1$

8. [7 points] Suppose w and q are continuous and invertible functions.
The table below shows many values of w and q^{-1} (the inverse of q).

s	-4.7	-3.3	-1.8	0.7	1.1	1.6	2.1	2.5	4.1	5.2
$w(s)$	4.1	2.5	1.4	0	-0.5	-1.8	-2	-3.1	-3.9	-4.7
$q^{-1}(s)$	-3.7	0.1	0.7	2.5	4.1	5.1	5.2	7.3	9.5	11.3

a. [2 points] Find $q^{-1}(w(-4.7))$.

Solution:
$q^{-1}(w(-4.7)) = q^{-1}(4.1) = 9.5$

Answer: 9.5

b. [2 points] Find $w(q(0.7))$.

Solution:
$w(q(0.7)) = w(-1.8) = 1.4$

Answer: 1.4

c. [3 points] Find the average rate of change of $q(x)$ between $x = 0.7$ and $x = 5.2$.
Be sure to show your work.

Solution: Average rate of change $= \frac{q(5.2) - q(0.7)}{5.2 - 0.7} = \frac{2.1 - (-1.8)}{4.5} = \frac{3.9}{4.5}$.

Answer: $\frac{3.9}{4.5} \approx 0.8667$

9. [12 points] Imtiyaz, Jacinta, and Katica, three food truck owners in San Francisco, gather to discuss the recent performances of their businesses. You may assume that all months have the same length. *Throughout this problem, be sure to show your work/reasoning carefully.*

a. [6 points] Jacinta feels that business has been slowing since she opened 6 months ago. She notes that she earned a total profit of \$2470 during her 3rd month of business and a total profit of \$1729 during her 5th month of business.

(i) Based on the data for months 3 and 5, if the profit of Jacinta's food truck were modeled by a linear function, what would the model predict her profit during her 9th month of business to be?

Solution: Linear functions have constant differences in output values for equally spaced input values. Profit changes by -741 (decreases by 741) dollars over two months so it changes by -1482 (decreases by 1482) dollars over four months.

Answer: \$247

(ii) Based on this data, if the profit of Jacinta's food truck were instead modeled by an exponential function, what would the model predict her profit during her 9th month of business to be?

Solution: Exponential functions have constant ratios in output values for equally spaced input values. Ratio of profit two months apart is 0.7, so it the is 0.7^2 for profit four months apart.

Answer: \$847.21

b. [3 points] Imtiyaz says that his profit for his 8th month of business was 45% higher than his profit in his 3rd month of business. If Imtiyaz's monthly profit increases by the same percentage every month, by what percent does it increase each month?

Solution: If a is the monthly growth factor, we know $a^5 = 1.45$, so $a = 1.45^{0.2}$. So the monthly percent growth is $1.45^{0.2} - 1 \approx 0.0771 = 7.71\%$.

Answer: about 7.71 percent

c. [3 points] Katica is really excited because her profit during her 12th month of business was 50% higher than in her 2nd month of business. If her profit is growing exponentially (as she hopes it is), in what month will Katica's profit be three times what it was in the 8th month? Round your answer to the nearest month.

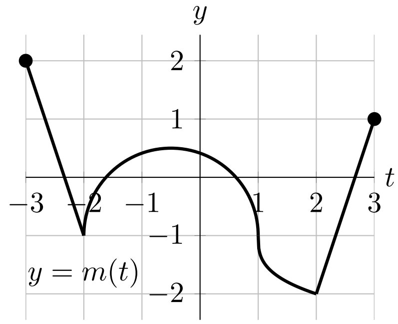
Solution: The monthly growth factor $a = 1.5^{0.1}$ (from $a^{10} = 1.5$).

Solving for $a^{t-8} = 3$ (from $a^t = 3 \cdot a^8$) gives $t - 8 = \dfrac{\ln(3)}{0.1\ln(1.5)}$.

So $t = 8 + 10\dfrac{\ln(3)}{\ln(1.5)} \approx 8 + 27.1 = 35.1$.

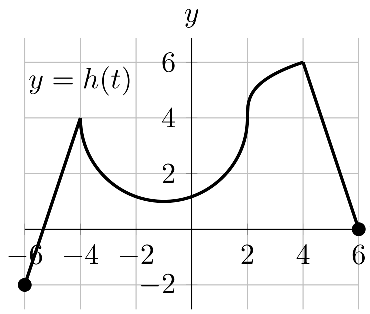
Answer: **during the 35th month**

10. [8 points]
The entire graph of a function m is shown on the right. Use this graph to answer the questions in parts **a.** and **b.** below.



Note that the scales on the axes of the graphs on this page are not all the same.

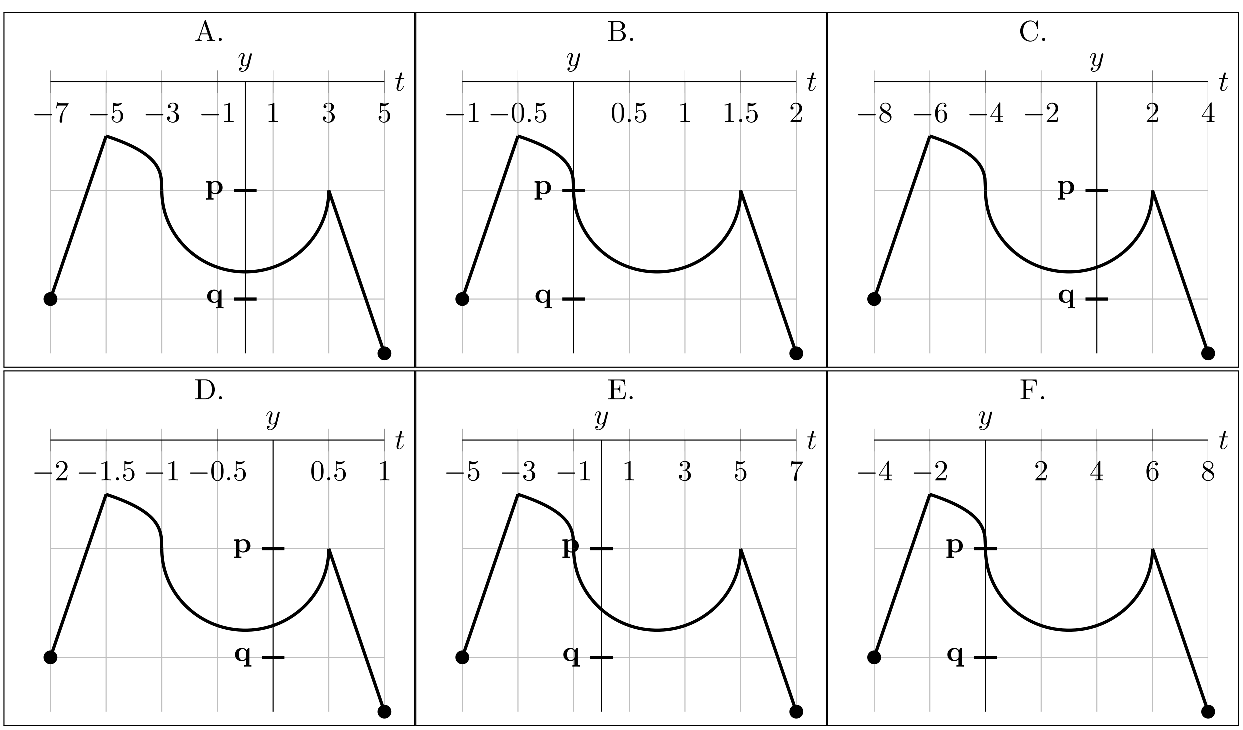
a. [4 points]
The graph of a function h is shown on the right. It is a transformation of the graph of m. Write a formula for $h(t)$ in terms of m and t.



Answer: $h(t) =$ $-2m(\frac{1}{2}t) + 2$

b. [4 points] Determine which one of the graphs A–F below is the graph of $y = -m(-2t + 1) - 3$. Then find the values of **p** and **q** shown on the graph you chose.

To recieve credit, you must circle an option (A–F) next to the word "Answer" below and write your values of **p** and **q** in the spaces provided.



Remember: to receive credit on this problem, you must circle one option below and write your values of **p** and **q** in the spaces provided.

Answer: A [B] C D E F

p = -2 and **q** = -4