Math 115 — Second Midterm — November 14, 2016

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	10	
3	7	
4	8	
5	10	
6	14	

Problem	Points	Score
7	10	
8	11	
9	12	
10	4	
11	6	
Total	12	

1. [8 points] The table below gives several values for the function f and its derivative f'. You may assume that f is invertible and differentiable.

w	-2	-1	0	1	2
$f(w)$	1	0	-2	-3	-5
$f'(w)$	-3	-1.5	-0.5	0	-4

For each of the parts below, find the exact value of the given quantity. If there is not enough information provided to find the value, write NOT ENOUGH INFO. If the value does not exist, write DOES NOT EXIST. You are not required to show your work on this problem. However, limited partial credit may be awarded based on work shown.

a. [2 points] Let $h(w) = \dfrac{f(w)}{6+w}$. Find $h'(-2)$.

Solution:
$$h'(w) = \frac{f'(w)(6+w) - f(w)\cdot 1}{(6+w)^2}$$
$$h'(-2) = \frac{(-3)\cdot(4) - 1}{4^2}$$

Answer: $h'(-2) = \dfrac{-13}{16} = -0.8125$

b. [2 points] Let $k(w) = 3^{f(2w)}$. Find $k'(-1)$.

Solution: $k'(w) = \ln(3)\cdot 3^{f(2w)}\cdot f'(2w)\cdot 2$
$k'(-1) = \ln(3)\cdot 3^{f(-2)}\cdot f'(-2)\cdot 2 = \ln(3)\cdot 3^1\cdot(-3)\cdot 2$

Answer: $k'(-1) = -18\ln(3)$

c. [2 points] Let $p(w) = f(f(-w+1))$. Find $p'(1)$.

Solution: $p'(w) = f'(f(-w+1))\cdot f'(-w+1)\cdot(-1)$
$p'(1) = f'(f(0))\cdot f'(0)\cdot(-1) = f'(-2)\cdot(-0.5)\cdot(-1) = (-3)\cdot(-0.5)\cdot(-1)$

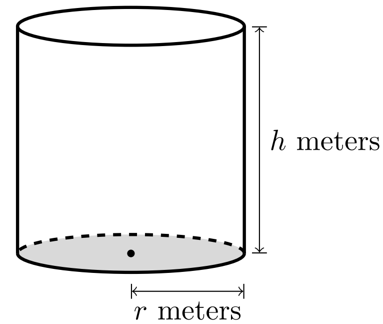
Answer: $p'(1) = -\dfrac{3}{2}$

d. [2 points] Let $r(w) = w\cdot(f(w))^2$. Find $r'(2)$.

Solution: $r'(w) = 1\cdot(f(w))^2 + w\cdot 2\cdot f(w)\cdot f'(w)$
$r'(2) = 1\cdot(f(2))^2 + 2\cdot 2\cdot f(2)\cdot f'(2) = 1\cdot(-5)^2 + 2\cdot 2\cdot(-5)\cdot(-4)$

Answer: $r'(2) = 105$

2. [10 points] Suma is making cylindrical paper cups that will be used to serve milkshakes at Qabil's Creamery. She rolls paper into a cylinder and then attaches it to the base. The thicker material that she uses for the base costs \$4.30 per square meter, and the lighter material that she uses for the vertical part of the cup costs \$2.20 per square meter. The radius of the circular base is r meters, and the height of the cup is h meters, as shown in the diagram on the right.
It may be helpful to know that the surface area of the vertical portion of the cup is $2\pi rh$.
Note: The top of the cup is left open.



Throughout this problem, assume that the material that Suma uses to make one paper cup costs \$0.12.

a. [4 points] Find a formula for h in terms of r.

Solution: The area of the vertical portion of the cup is $2\pi rh$ square meters, so the cost for the material for the vertical portion of one cup is $(2.20)(2\pi rh)$ dollars. Since the base of the cup is circular, its area is πr^2 square meters, and the cost for the material for the base of one cup is $(4.30)(\pi r^2)$ dollars. So the material that Suma uses to make one cup costs a total of $(2.20)(2\pi rh) + (4.30)(\pi r^2)$ dollars.
Therefore, we have $(2.20)(2\pi rh) + (4.30)(\pi r^2) = 0.12.$
Solving for h we find that $h = \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$.

Answer: $h = \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$

b. [2 points] Let $V(r)$ be the volume (in cubic meters) of the cup that Suma makes given that the material for the cup costs \$0.12 and the radius of the cup is r meters. Find a formula for $V(r)$. The variable h should not appear in your answer.
(Note: This is the function that Suma would use to find the value of r maximizing the volume of the cup, but you should not do the optimization in this case.)

Solution: Since Suma's cup is a cylinder, its volume is $\pi r^2 h$. So using what we found in part **a.** above, we see that $V(r) = \pi r^2 \cdot \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$ which simplifies to $V(r) = \dfrac{r(0.12 - 4.3\pi r^2)}{4.4}$.

Answer: $V(r) = (\pi r^2) \cdot \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$ **or** $\dfrac{r(0.12 - 4.3\pi r^2)}{4.4}$

c. [4 points] In the context of this problem, what is the domain of $V(r)$?

Solution: Since r is a length, it cannot be negative. Note also that if $r = 0$, then the cost of the materials for the cup would be 0 dollars (rather than \$0.12), so r must be positive. The height h also cannot be negative, and as h decreases, r increases. Therefore, r can certainly not be greater than when $h = 0$, in which case $4.3(\pi r^2) = 0.12$, so $r = \sqrt{\dfrac{0.12}{4.3\pi}}$ (since r must be positive). In this case, the cup would have height 0 and thus hold no milkshake, so we may choose to exclude this endpoint of the domain.

Answer: **the interval** $\left(0, \sqrt{\dfrac{0.12}{4.3\pi}}\right)$

3. [7 points] Consider the curve $\mathcal{D}$ defined by the equation

$$x^2y(1-y) = 9.$$

Note that the curve $\mathcal{D}$ satisfies $\dfrac{dy}{dx} = \dfrac{2xy(y-1)}{x^2(1-2y)}$.

a. [4 points] Exactly one of the following points (x, y) lies on the curve $\mathcal{D}$. Circle that one point.

$(0.9, 10)$ $(1, -8)$ $(3, 9)$ $(9, 3)$ $\boxed{(10, 0.9)}$

Then find an equation for the tangent line to the curve $\mathcal{D}$ at the point you chose.

Solution: At the point $(0.9, 10)$, the slope of the tangent line is

$$\frac{2\cdot 10\cdot 0.9\cdot(0.9-1)}{100\cdot(1-2\cdot 0.9)} = \frac{1.8}{80} = \frac{9}{400} = 0.0225.$$

Answer: $y =$ $0.9 + \dfrac{1.8}{80}(x-10)$ $\quad (= 0.625 + 0.0225x)$

b. [3 points] Find all points on the curve $\mathcal{D}$ where the slope of the curve is undefined. Give your answers as ordered pairs. Write NONE if there are no such points.

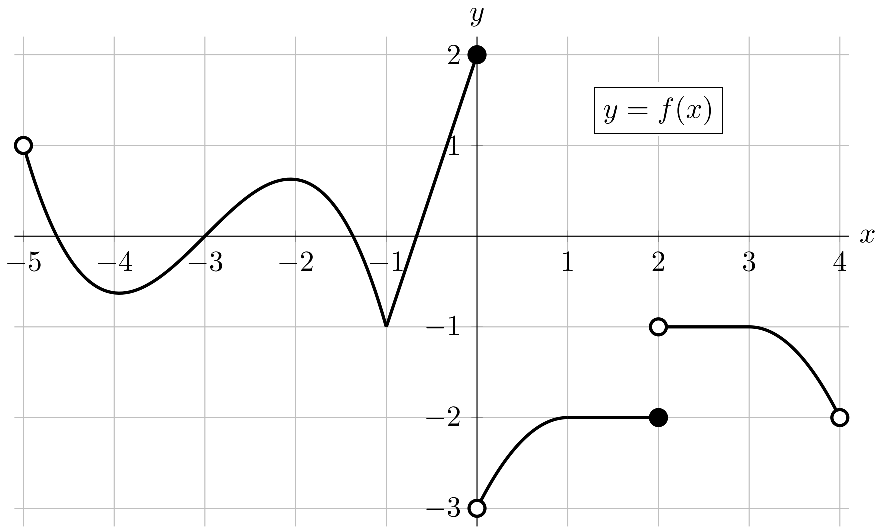
Solution: The slope is undefined for points on $\mathcal{D}$ when the denominator of $\dfrac{dy}{dx}$ is 0. This happens when $x^2(1-2y) = 0$, so $x = 0$ or $y = \frac{1}{2}$.

When $x = 0$, we know that $x^2y(1-y) = 0$ (rather than 9), so there are no such points on the curve $\mathcal{D}$.

When $y = \frac{1}{2}$, the equation for the curve gives $x^2 \cdot \frac{1}{2}(1-\frac{1}{2}) = 9$. So $x^2 = 36$ and therefore $x = \pm 6$. This results in the two points $(6, \frac{1}{2})$ and $(-6, \frac{1}{2})$.

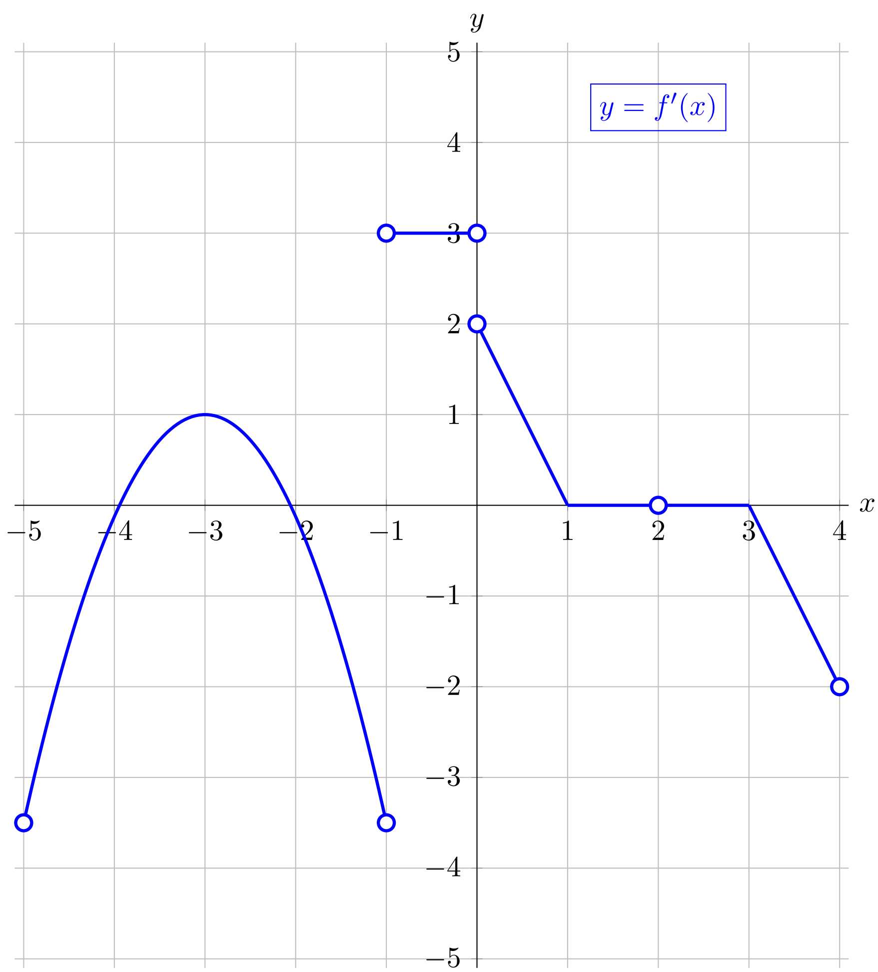
Answer: $(x, y) =$ $(6, \frac{1}{2}), (-6, \frac{1}{2})$

4. [8 points] The graph of a function f is shown below.



On the axes below, sketch a graph of $f'(x)$ (the derivative of the function $f(x)$) on the interval $-5 < x < 4$. Be sure that you pay close attention to each of the following:

- where f' is defined
- the value of $f'(x)$ near each of $x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4$
- the sign of f'
- where f' is increasing/decreasing/constant



5. [10 points] As a software engineer, Tendai spends many hours every day writing code. Let $w(t)$ be a function that models the number of lines of code that Tendai writes in a day if he works t hours that day. Tendai works at least one hour and at most 18 hours each day. A formula for $w(t)$ is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \le t \le 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \le 18. \end{cases}$$

a. [8 points] Find the values of t that minimize and maximize $w(t)$ on the interval $[1, 18]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE if appropriate.

Solution: Note that w is continuous at $t = 3$, since $\lim_{t\to 3^-} w(t) = \lim_{t\to 3^+} w(t) = 66$, so we may use the Extreme Value Theorem.
We find

$$w'(t) = \begin{cases} -4t + 28 & \text{if } 1 < t < 3 \\ -t + 9 & \text{if } 3 < t < 18. \end{cases}$$

The first expression is 0 when $t = 7$, but since this isn't in the domain of that piece, it is not a critical point. The second expression is 0 when $t = 9$.
Since both of these are polynomials, we don't have to worry about the derivative not existing on these open intervals. However, since $-4 \cdot 3 + 28 = 16$ and $-3 + 9 = 6$ are not equal, w' is not defined at 3, so $t = 3$ is also a critical point.

Computing $w(t)$ at each critical point and the endpoints gives:

t	1	3	9	18
$w(t)$	26	66	84	43.5

By the Extreme Value Theorem, we therefore find that $w(t)$ attains its maximum value at $t = 9$ and its minimum at $t = 1$.

Answer: global max(es) at $t =$ 9

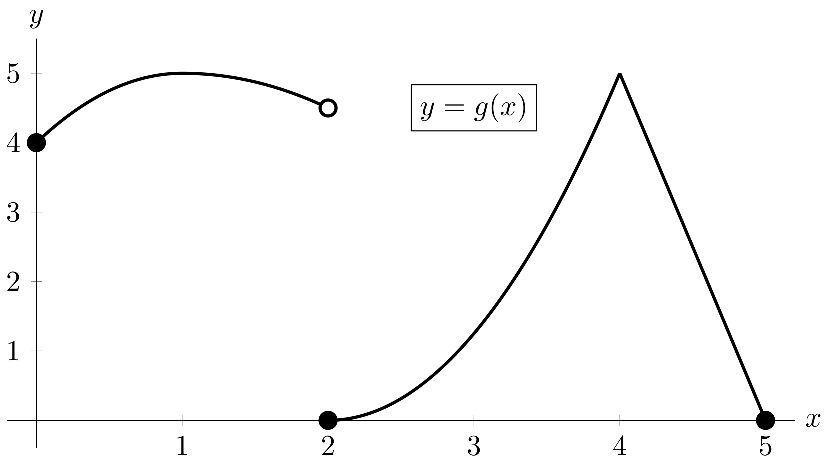
Answer: global min(s) at $t =$ 1

b. [2 points] What is the largest number of lines of code that Tendai can expect to write in a day according to this model?

Solution: From part **a.** we see that the maximum value of w is $w(9) = 84$. So according to this model, the largest number of lines of codes that Tendai can expect to write in a day is 84.

Answer: 84

6. [14 points] The entire graph of a function $g(x)$ is shown below. Note that the graph of $g(x)$ has a horizontal tangent line at $x = 1$ and a sharp corner at $x = 4$.



For each of the questions below, circle <u>all</u> of the available correct answers.
(Circle NONE OF THESE if none of the available choices are correct.)

a. [2 points] At which of the following values of x does $g(x)$ appear to have a critical point?

$\boxed{x = 1}$ $\boxed{x = 2}$ $x = 3$ $\boxed{x = 4}$ NONE OF THESE

b. [2 points] At which of the following values of x does $g(x)$ attain a local maximum?

$\boxed{x = 1}$ $x = 2$ $x = 3$ $\boxed{x = 4}$ NONE OF THESE

c. [6 points] Let $L(x)$ be the local linearization of $g(x)$ near $x = 3$. Circle all of the statements that are true.

$L(3) > g(3)$	$L(2.5) > g(2.5)$	$L(0) > g(0)$
$\boxed{L(3) = g(3)}$	$L(2.5) = g(2.5)$	$L(0) = g(0)$
$L(3) < g(3)$	$\boxed{L(2.5) < g(2.5)}$	$\boxed{L(0) < g(0)}$
$L'(3) > g'(3)$	$\boxed{L'(2.5) > g'(2.5)}$	$\boxed{L(5) > g(5)}$
$\boxed{L'(3) = g'(3)}$	$L'(2.5) = g'(2.5)$	$L(5) = g(5)$
$L'(3) < g'(3)$	$L'(2.5) < g'(2.5)$	$L(5) < g(5)$

NONE OF THESE

d. [2 points] On which of the following intervals does $g(x)$ satisfy the hypotheses of the Mean Value Theorem?

$[0, 2]$ $[0, 4]$ $[3, 5]$ $\boxed{[4, 5]}$ NONE OF THESE

e. [2 points] On which of the following intervals does $g(x)$ satisfy the conclusion of the Mean Value Theorem?

$[0, 2]$ $\boxed{[0, 4]}$ $[3, 5]$ $\boxed{[4, 5]}$ NONE OF THESE

7. [10 points] Suppose $f(x)$ is a continuous function defined for all real numbers whose derivative and second derivative are given by

$$f'(x) = \arctan\left(\frac{(2x-3)^2}{(x+1)^3}\right) \qquad \text{and} \qquad f''(x) = \frac{(x+1)^3(2x-3)(13-2x)}{(x+1)\left[(x+1)^6+(2x-3)^4\right]}.$$

a. [5 points] Find all critical points of $f(x)$ and all values of x at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: First we find the critical points, which occur when $f'(x) = 0$ or $f'(x)$ does not exist. Since $f'(x) = 0$ when $x = \frac{3}{2}$ and $f'(x)$ is not defined when $x = -1$, the two critical points are $x = -1$ and $x = \frac{3}{2}$.
Next we must determine whether there is a local min, local max, or neither at each critical point. The second derivative test is inconclusive, because $f''(-1)$ does not exist and $f''(\frac{3}{2}) = 0$. So we must use the first derivative test. Notice that arctan preserves signs (i.e., $\arctan(x)$ is positive when x is positive and $\arctan(x)$ is negative when x is negative) so we only need to check the sign of $\left(\frac{(2x-3)^2}{(x+1)^3}\right)$. The factor $(2x-3)^2$ is always positive, while $(x+1)^3$ is negative when $x < -1$ and positive when $x > -1$. This gives us the resulting signs:

Interval	$x < -1$	$-1 < x < \frac{3}{2}$	$x > \frac{3}{2}$
Sign of $f'(x)$	$\frac{+}{-} = -$	$\frac{+}{+} = +$	$\frac{+}{+} = +$

So $f(x)$ has a local minimum at $x = -1$ and no local maxima.

Answer: Critical point(s) at $x =$ $-1, \frac{3}{2}$

Local max(es) at $x =$ **None** Local min(s) at $x =$ -1

b. [5 points] Find the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: First we find the candidate inflection points, which occur when $f''(x) = 0$ or $f''(x)$ does not exist. We can see that $f''(x) = 0$ at $x = \frac{3}{2}$ and $\frac{13}{2}$ and that $f''(x)$ is undefined when $x = -1$. To determine whether these are actually inflection points (where concativty changes), we must test the sign of the second derivative on either side of each of the points. We find the following:

Interval	$x < -1$	$-1 < x < \frac{3}{2}$	$\frac{3}{2} < x < \frac{13}{2}$	$x > \frac{13}{2}$
Sign of $f'(x)$	$\frac{-\cdot-\cdot+}{-\cdot+} = -$	$\frac{+\cdot-\cdot+}{+\cdot+} = -$	$\frac{+\cdot+\cdot+}{+\cdot+} = +$	$\frac{+\cdot+\cdot-}{+\cdot+} = -$

The inflection points of $f(x)$ occur at the points where the sign of the second derivative changes, that is, at $x = \frac{3}{2}$ and $x = \frac{13}{2}$.

Answer: Inflection point(s) at $x =$ $\frac{3}{2}, \frac{13}{2}$

8. [11 points] Pepukai is studying the effect of the availability of water on the fruit productivity of Michigan apple trees. She observes that Michigan apple trees produce very few apples if they have too little water. She determines a function $p(w)$ that models the total weight, in pounds, of all the apples that an average Michigan apple tree produces in a season when it is watered with w gallons of water every week. The domain of p is $[5, 40]$. Some values of the function p and its derivative p' are shown in the table below.

w	10	15	20	25	30
$p(w)$	25	96	118	129	135
$p'(w)$	96	13	4	2	1

The function p is invertible and the functions p, p', and p^{-1} are all differentiable. Furthermore, the function p' is always decreasing.

a. [3 points] Find $(p^{-1})'(96)$.

Solution: $(p^{-1})'(96) = \frac{1}{p'(p^{-1}(96))} = \frac{1}{p'(15)} = \frac{1}{13} \approx 0.07692$

Answer: $(p^{-1})'(96) =$ $\frac{1}{13}$

b. [2 points] Circle the one statement that is best supported by the equation

$$(p^{-1})'(10) = 0.01.$$

A. $\boxed{\text{To increase the total weight of apples produced in a season by an average Michigan apple tree from 10 pounds to 11 pounds, the tree should be watered with about 0.01 additional gallons of water every week.}}$

B. If an average Michigan apple tree produces 10 pounds of apples in a season, watering the tree with 1 extra gallon every week increases the total weight of apples produced by the tree in a season by about 0.01 pounds.

C. If the amount of water that an average Michigan apple tree is watered with increases from 10 gallons every week to 10.1 gallons every week, the total weight of apples produced by the tree in a season increases by about 10 pounds.

D. If the amount of water that an average Michigan apple tree is watered with increases from 10 gallons every week to 10.1 gallons every week, the total weight of apples produced by the tree in a season increases by about 0.001 pounds.

c. [3 points] Write a formula for $g(w)$, the tangent line approximation to $p(w)$ near $w = 15$.

Answer: $g(w) =$ $13(w - 15) + 96$

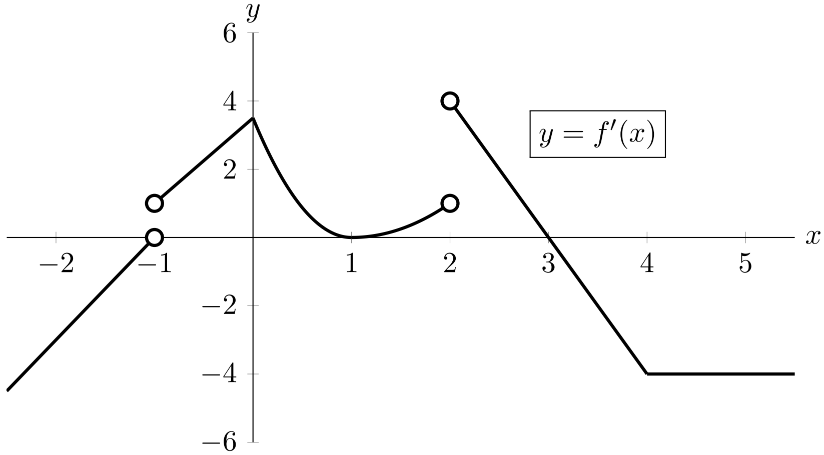
d. [3 points] Does the tangent line approximation $g(w)$ give an underestimate or overestimate of the value of $p(w)$ at $w = 18$? Justify your answer.

Circle one: underestimate $\boxed{\text{overestimate}}$ CANNOT BE DETERMINED

Justification:

Solution: Since p' is always decreasing, the function p is always concave down. So the tangent line at $w = 15$ lies above the graph of p and therefore the tangent line approximation gives an overestimate of the value of $p(w)$ at $w = 18$.

9. [12 points] The graph of a portion of the <u>derivative</u> of a function $f(x)$ is given below. Assume that the domain of f is all real numbers, and that f is continuous on the entire interval $[-2, 5]$.



Use the graph above to answer the following questions. For each question, circle <u>all</u> of the available correct answers.
(Circle NONE OF THESE if none of the available choices are correct.)

a. [2 points] At which of the following values of x does $f(x)$ appear to have a critical point?

$x = 0$ $\boxed{x = 1}$ $\boxed{x = 2}$ $\boxed{x = 3}$ $x = 4$ NONE OF THESE

b. [2 points]
At which of the following values of x does $f'(x)$ appear to have a critical point?

$\boxed{x = 0}$ $\boxed{x = 1}$ $x = 3$ $\boxed{x = 4}$ NONE OF THESE

c. [2 points] At which of the following values of x does $f(x)$ attain a local extremum?

$\boxed{x = -1}$ $x = 0$ $x = 1$ $\boxed{x = 3}$ NONE OF THESE

d. [2 points] At which of the following values of x does $f(x)$ attain a global maximum on the interval $[-1, 3]$?

$x = -1$ $x = 0$ $x = 1$ $x = 2$ $\boxed{x = 3}$ NONE OF THESE

e. [2 points] At which of the following values of x does $f(x)$ have an inflection point?

$x = -1$ $\boxed{x = 0}$ $\boxed{x = 1}$ $\boxed{x = 2}$ $x = 3$ NONE OF THESE

f. [2 points] For which of the following intervals is $f(x)$ concave up on the <u>entire</u> interval?

$\boxed{-1 < x < 0}$ $0 < x < 1$ $\boxed{1 < x < 2}$ $2 < x < 4$ NONE OF THESE

10. [4 points] Let a and b be constants. Consider the curve $\mathcal{C}$ defined by the equation

$$\cos(ax) + by\ln(x) = 3 + y^3.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution: We use implicit differentiation.

$$\begin{aligned} \frac{d}{dx}(\cos(ax) + by\ln(x)) &= \frac{d}{dx}(3 + y^3) \\ -a\sin(ax) + \frac{by}{x} + b\ln(x)\frac{dy}{dx} &= 3y^2\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)} \end{aligned}$$

Answer: $\frac{dy}{dx} = \boxed{\frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)}}$

11. [6 points] Let $h(x) = x^x$. For this problem, it may be helpful to know the following formulas:

$$h'(x) = x^x(\ln(x) + 1) \qquad \text{and} \qquad h''(x) = x^x\left(\frac{1}{x} + (\ln(x)+1)^2\right).$$

a. [2 points] Write a formula for $p(x)$, the local linearization of $h(x)$ near $x = 1$.

Solution: $h(1) = 1$ and $h'(1) = 1^1(\ln(1)+1) = 1$, so $p(x) = 1 + 1\cdot(x-1) = x$.

Answer: $p(x) =$ x

b. [4 points] Write a formula for $u(x)$, the quadratic approximation of $h(x)$ at $x = 1$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$.)

Solution: $h''(1) = 1(1 + (0+1)^2) = 2$, so $u(x) = 1 + (x-1) + \frac{2}{2}(x-1)^2 = x^2 - x + 1$.

Answer: $u(x) =$ $1 + (x-1) + (x-1)^2$ $(= x^2 - x + 1)$