Math 115 — Final Exam — December 19, 2016

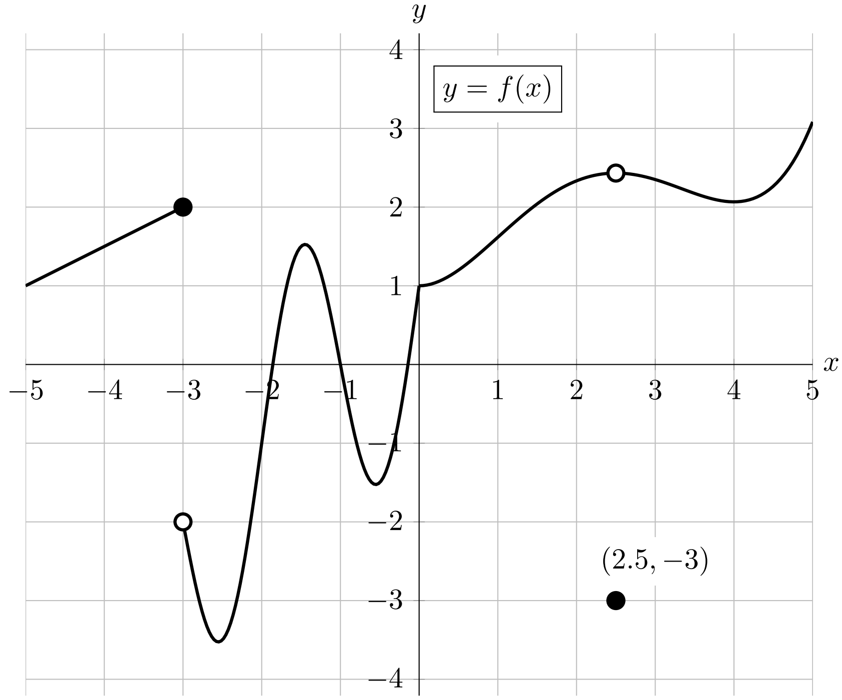
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 14 pages including this cover. There are 12 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	9	
3	7	
4	10	
5	10	
6	4	

Problem	Points	Score
7	5	
8	9	
9	9	
10	10	
11	10	
12	8	
Total	100	

1. [9 points] A portion of the graph of a function f is shown below.



Throughout this problem, you do not need to explain your reasoning.

For each of parts **a.**– **d.** below, circle all of the listed values satisfying the given statement. If there are no such values, circle NONE.

a. [2 points] For which of the following values of c is $\lim_{x\to c^-} f(x) = f(c)$?

$\boxed{c=-3}$ $\boxed{c=-1}$ $\boxed{c=0}$ $\boxed{c=1}$ $c=2.5$ NONE

b. [2 points] For which of the following values of c is $f(x)$ continuous at $x=c$?

$c=-3$ $\boxed{c=-1}$ $\boxed{c=0}$ $\boxed{c=1}$ $c=2.5$ NONE

c. [2 points] For which of the following values of c does $f(x)$ appear to be differentiable at $x=c$?

$c=-3$ $\boxed{c=-1}$ $c=0$ $\boxed{c=1}$ $c=2.5$ NONE

d. [3 points] Consider the quantities defined as follows:

I. The number 0. II. $f(1)$. III. $\int_{-1}^{1} f(x)\,dx$.

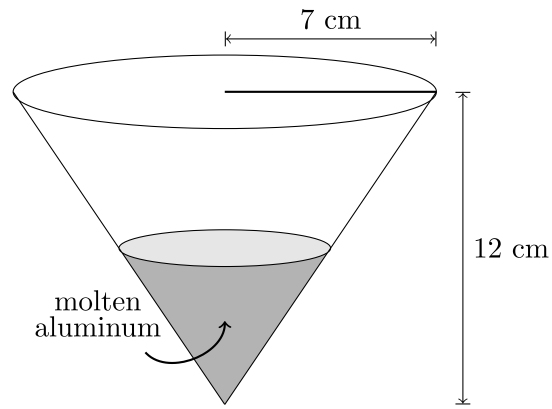
IV. The left-hand Riemann sum with 2 equal subintervals for $\int_{-1}^{1} f(x)\,dx$.

V. The right-hand Riemann sum with 2 equal subintervals for $\int_{-1}^{1} f(x)\,dx$.

Rank the quantities in order from least to greatest by filling in the blanks below with the options I–V. You do not need to show your work.

I < **III** < **IV** < **II** < **V**

2. [9 points] Uri is filling a cone with molten aluminum. The cone is upside-down, so the "base" is at the top of the cone and the vertex at the bottom, as shown in the diagram. The base is a circular disk with radius 7 cm and the height of the cone is 12 cm.
Recall that the volume of a cone is $\frac{1}{3}Ah$, where A is the area of the base and h is the height of the cone (i.e., the vertical distance from the vertex to the base). (Note that the diagram may not be to scale.)



a. [3 points] Write a formula in terms of h for the volume V of molten aluminum, in cm^3, in the cone if the molten aluminum in the cone reaches a height of h cm.

Solution: Let r be the radius of the top surface of the molten aluminum. Using similar triangles, we see $r = \dfrac{7h}{12}$. Since the top surface of the molten aluminum is a circular disk, its area is πr^2.
So $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi\left(\dfrac{7}{12}\right)^2 h^3 \left(= \dfrac{49\pi}{432}h^3\right)$.

Answer: $V = \dfrac{\pi}{3}\left(\dfrac{7}{12}\right)^2 h^3$

b. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm^3/sec, at which Uri is pouring molten aluminum into the cone at that moment?

Solution: We differentiate $V = \dfrac{\pi}{3}\left(\dfrac{7}{12}\right)^2 h^3$ with respect to h to get $\dfrac{dV}{dh} = \pi\left(\dfrac{7}{12}\right)^2 h^2$.
So, $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = \pi\left(\dfrac{7}{12}\right)^2 h^2 \cdot \dfrac{dh}{dt}$.
(Alternatively, differentiating both sides of the equation $V = \dfrac{\pi}{3}\left(\dfrac{7}{12}\right)^2 h^3$ with respect to t results in the same formula for $\dfrac{dV}{dt}$.)
We are given that $\left.\dfrac{dh}{dt}\right|_{h=11} = 3$, so we find
$\left.\dfrac{dV}{dt}\right|_{h=11} = \pi\left(\dfrac{7}{12}\right)^2 11^2 \cdot 3 = \dfrac{17787\pi}{144} = \dfrac{5929\pi}{48} \approx 388.052.$
Thus at the moment when the molten aluminum in the cone has reached a height of 11 cm, Uri is pouring molten aluminum into the cone at a rate of $\dfrac{5929\pi}{48}$ (or about 388.052) cm^3/sec

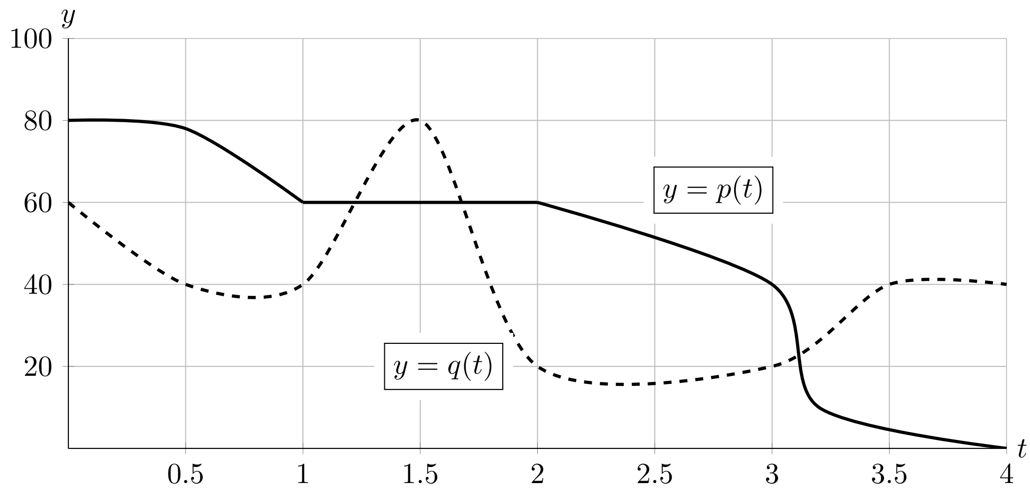
Answer: $\pi\left(\dfrac{7}{12}\right)^2 11^2 \cdot 3 = \dfrac{5929\pi}{48} \approx 388.052$

c. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm^2/sec, at which the area of the top surface of the molten aluminum is increasing at that moment?

Solution: Let A be the area of the top surface of the molten aluminum.
Since $A = \pi r^2 = \pi \left(\frac{7}{12}\right)^2 h^2$, we see that $\frac{dA}{dh} = 2\pi \left(\frac{7}{12}\right)^2 h$.
So, $\frac{dA}{dt} = \frac{dA}{dh}\frac{dh}{dt} = 2\pi \left(\frac{7}{12}\right)^2 h \cdot \frac{dh}{dt}$.
(Alternatively, differentiating both sides of the equation $A = \pi \left(\frac{7}{12}\right)^2 h^2$ with respect to t results in the same formula for $\frac{dA}{dt}$.)
We are given that $\left.\frac{dh}{dt}\right|_{h=11} = 3$, so we find
$\left.\frac{dA}{dt}\right|_{h=11} = 2\pi \left(\frac{7}{12}\right)^2 11 \cdot 3 = \frac{3234\pi}{144} = \frac{539\pi}{24} \approx 70.5549$.
Thus at the moment when the molten aluminum in the cone has reached a height of 11 cm, the area of the top surface of the molten aluminum is increasing at a rate of $\frac{539\pi}{24}$ (or about 70.5549) cm^2/sec.

Answer: $22\pi \left(\frac{7}{12}\right)^2 \cdot 3 = \frac{539\pi}{24} \approx 70.5549$

3. [7 points] At the cider mill, Xanthippe makes donuts fastest when she isn't distracted by customers. The rate, in donuts per hour, at which Xanthippe makes donuts t hours after 7 am is modeled by the function $p(t)$. Customers purchase donuts during their visit to the cider mill. The rate, in donuts per hour, at which customers purchase donuts t hours after 7 am is modeled by the function $q(t)$. The graphs of $y = p(t)$ (solid) and $y = q(t)$ (dashed) are shown below. Assume that at 7 am, Xanthippe begins with no donuts in stock.



a. [2 points] At what rate, in donuts per hour, is the number of donuts in stock (donuts produced but not yet sold) increasing/decreasing at 8:30 am? Be sure to circle one of INCREASING or DECREASING.

Solution: At $t = 1.5$, $p(t) - q(t) = -20$. The rate at which donuts are being sold exceeds the rate at which the donuts are being produced at a rate of 20 donuts/hr. Therefore, the number of donuts in stock is decreasing at a rate of 20 donuts/hr.

Answer: INCREASING $\boxed{\text{DECREASING}}$ at a rate of **20 donuts/hr**

b. [2 points] Write an expression involving p and q for the number of donuts in stock at 10 am. Your answer may involve definite integrals. Do not give approximations.

Solution: $p(t) - q(t)$ is the rate at which the number of donuts in stock is changing t hours after 7 am. By the fundamental theorem of calculus, $\int_0^3 p(t) - q(t)\,dt$ is the change in the number of donuts in stock between 7 am and 10 am. Since there were no donuts in stock at at 7 am, this is the number of donuts in stock at 10 am.

Answer: $\displaystyle\int_0^3 p(t) - q(t)\,dt$

c. [3 points] Xanthippe stops making donuts at 11 am. Assume that after 11 am, customers continue to purchase donuts at a constant rate of 40 donuts per hour until all of Xanthippe's donuts are sold out. Write an expression for the number of hours, starting at 11 am, that it takes for all her donuts to be sold out. Your answer may involve definite integrals. Do not give approximations.

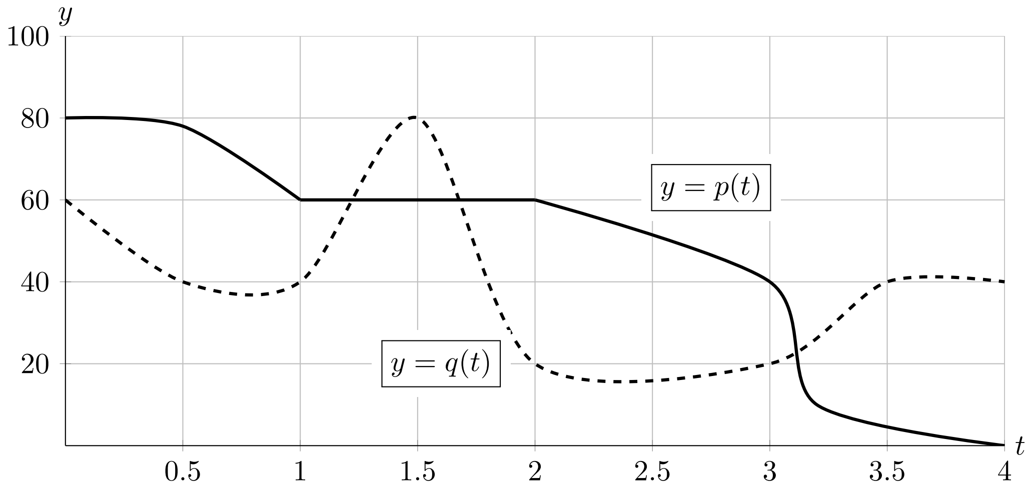
Solution: The number of donuts in stock at 11 am is $\int_0^4 p(t) - q(t)\,dt$. When s hours have passed after 11 am, $40s$ donuts have been sold (assuming all donuts were not already sold), so we want to find s such that $40s = \int_0^4 p(t) - q(t)\,dt$.

Answer: $\displaystyle\frac{1}{40}\int_0^4 p(t) - q(t)\,dt$

This problem continues the investigation of Xanthippe's donuts.

4. [10 points] For your convenience, the graphs of $p(t)$ and $q(t)$ are reprinted below. Recall:

- The rate, in donuts per hour, at which Xanthippe makes donuts t hours after 7 am is modeled by the function $p(t)$.
- The rate, in donuts per hour, at which customers purchase donuts t hours after 7 am is modeled by the function $q(t)$.
- Assume that at 7 am, Xanthippe begins with no donuts in stock.



a. [4 points] Estimate the total number of donuts produced by 10 am using a right-hand Riemann sum with two equal subintervals. Be sure to write down all the terms in your sum. Is your answer an underestimate or overestimate?

Solution: Each subinterval has width $\Delta t = 1.5$. Therefore, a right-hand Riemann sum with two equal subintervals is $\int_0^3 p(t)\,dt \approx p(1.5)\cdot 1.5 + p(3)\cdot 1.5 = 60\cdot 1.5 + 40\cdot 1.5$

Answer: donuts produced by 10 am $\approx$ **150**

This is an (circle one) OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$

b. [4 points] The number of donuts in stock t hours after 7 am is modeled by the function $s(t)$. Estimate the t-values for all critical points of $s(t)$ in the interval $0 < t < 4$, and estimate all values of t in the interval $0 < t < 4$ at which $s(t)$ has a local extremum. For each answer blank write NONE if appropriate. You do <u>not</u> need to justify your answers.

Solution: We know $s'(t) = p(t) - q(t)$. Since $p(t)$ and $q(t)$ are defined on $0 < t < 4$, we only need to find where $p(t) - q(t) = 0$. In other words, where $p(t) = q(t)$. From the graph, we can see that $s(t)$ goes from positive to negative at $t = 1.2$ and $t = 3.1$ and from negative to positive at $t = 1.7$.

Answer: Critical point(s) at $t =$ **1.2, 1.7, 3.1**

Local max(es) at $t =$ **1.2, 3.1** Local min(s) at $t =$ **1.7**

c. [2 points] At what time is the number of donuts that Xanthippe has in stock the greatest? Round your answer to the nearest half hour. You do <u>not</u> need to justify your answer.

Answer: **10:00 am**

5. [10 points] The table below gives several values of a function $q(u)$ and its first and second derivatives. Assume that all of $q(u)$, $q'(u)$, and $q''(u)$ are defined and continuous for all real numbers u.

u	0	1	2	3	4	5	6
$q(u)$	30	23	19	20	24	25	24
$q'(u)$	0	-6	-2	1	3	1	-2
$q''(u)$	-9	5	4	3	2	-5	0

Compute each of the following. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

a. [2 points] Compute $\displaystyle\int_5^2 q''(t)\,dt$.

Solution: $\int_5^2 q''(t)\,dt = q'(2) - q'(5) = -2 - 1 = -3.$

Answer: $\displaystyle\int_5^2 q''(t)\,dt =$ -3

b. [2 points] Compute $\displaystyle\int_1^5 (-2q''(u) + 2u)\,du$.

Solution: $\int_1^5 (-2q''(u)+2u)\,du = (-2q'(5)+5^2) - (-2q'(1)+1^2) = (-2+25) - (12+1) = 10.$

Answer: $\displaystyle\int_1^5 (-2q''(u) + 2u)\,du =$ 10

c. [2 points] Suppose that $q(u)$ is an even function. Compute $\displaystyle\int_{-5}^5 q(u)\,du$.

Solution: $\int_{-5}^5 q(u)\,du = 2\int_0^5 q(u)\,du$. This cannot be computed exactly.

Answer: $\displaystyle\int_{-5}^5 q(u)\,du =$ **not possible**

d. [2 points] Suppose that $q(u)$ is an even function. Compute $\displaystyle\int_{-5}^5 \left(q'(u) + 7\right)\,du$.

Solution: $\int_{-5}^5 (q'(u)+7)\,du = (q(5) + 7\cdot 5) - (q(-5) + 7\cdot(-5))$. Since $q(5) = q(-5)$, we have $\int_{-5}^5 (q'(u)+7)\,du = q(5) - q(-5) + 7\cdot 10 = 70$.

Answer: $\displaystyle\int_{-5}^5 \left(q'(u) + 7\right)\,du =$ 70

e. [2 points] Compute the average value of $-5q'(u)$ on the interval $[1, 4]$.

Solution: Average value $= \frac{1}{4-1}\int_1^4 -5q'(u)\,du = \frac{1}{3}\left[-5q(4) - (-5q(1))\right] = \frac{5}{3}\left[q(1) - q(4)\right] = \frac{-5}{3}$.

Answer: $\dfrac{-5}{3}$

6. [4 points] Formulas for a function $g(x)$ and its derivative $g'(x)$ are given below.

$$g(x) = (2-4x)e^{-x^2} \quad \text{and} \quad g'(x) = 4(2x+1)(x-1)e^{-x^2}.$$

Find all global extrema of $g(x)$ on the open interval $(0, \infty)$. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all global extrema. Write NONE if appropriate.

Solution: The critical points of g are at $x = \frac{-1}{2}$ and $x = 1$. The first is not in $(0, \infty)$.

Option 1: using a derivative test.

First derivative test: note that e^{-x^2} is always positive. When $0 < x < 1$, $2x+1$ is positive and $x-1$ is negative, so $g'(x) < 0$. When $x > 1$, all of the factors in $g'(x)$ are positive, so $g'(x) > 0$. This tells us that g has a local minimum at $x = 1$.

Second derivative test: $g''(x) = -4e^{-x^2}(1-6x-2x^2+4x^3)$, so $g''(1) = -4e^{-1}(1-6-2+4) = 12e^{-1} > 0$, so g has a local minimum at $x = 1$.

After using one of these tests to determine that $g(x)$ has a local minimum at $x = 1$, we need to say something more to answer the question about the location of global extrema.

Since there's only one critical point, and the endpoint $x = 0$ is not included in our interval, the local minimum at $x = 1$ is also the global minimum, and there is no global maximum.

Option 2: using limits

Note that $g(x)$ is a continuous function. We see that $\lim_{x \to 0} g(x) = g(0) = 2$, $g(1) = -2e^{-1}$, and $\lim_{x \to \infty} g(x) = 0$. Since $-2e^{-1} < 0$, we have a global minimum at $x = 1$ and no global maximum.

Answer: global max(es) at $x =$ **None**

global min(s) at $x =$ **1**

7. [5 points] Consider the family of functions given by $g(x) = x\ln(px^2 + q)$, for constants p and q. Find values of p and q so that the function has a local extremum at $(1, 2)$. Be sure to justify (using calculus) that your resulting function does have a local extremum at $(1, 2)$ and to determine the type of extremum. Leave your answers in exact form.
You may find the following information to be useful.

$$g'(x) = \ln\left(px^2 + q\right) + \frac{2px^2}{px^2 + q} \quad \text{and} \quad g''(x) = \frac{2px(px^2 + 3q)}{\left(px^2 + q\right)^2}$$

Solution: In order to have a local extremum at $(1, 2)$, we must have that $g(1) = 2$ and $g'(1) = 0$ or does not exist. So we need

$$1 \cdot \ln(p + q) = 2 \text{ and } \ln(p + q) + \frac{2p}{p + q} = 0.$$

(Note that for the first equation to be true, $p + q > 0$, so $g'(1)$ exists.)
Since $\ln(p + q) = 2$, we have $p + q = e^2$, so the second equation reduces to $2 + \frac{2p}{e^2} = 0$. This is true when $p = -e^2$. Plugging this back into the first equation, we find $q = 2e^2$.
To find whether this is a local min or max, we can plug these values for p and q into the second derivative and evaluate it when $x = 1$. This gives $\frac{-2e^2(-e^2 + 3 \cdot 2e^2)}{e^4}$. The denominator of this is positive, while the numerator is negative, giving us $g''(1) < 0$, so we have a local max at $(1, 2)$.

Answer: $p =$ $-e^2$ and $q =$ $2e^2$

Circle one: $\boxed{\text{LOCAL MAXIMUM}}$ LOCAL MINIMUM

8. [9 points] Zoltan is undergoing an anti-aging skin treatment that involves a machine that uses electrical current to deliver medicine through the skin. During a treatment session, the total amount of medicine that has been absorbed by the skin is a function of the total electrical charge that has entered the skin.
A particular treatment session begins before noon and ends after 12:30 pm, and at noon, Zoltan has already absorbed 4 mg of the medicine.

- Let $m(c)$ be the total amount of medicine, in mg, that has been absorbed when a total electrical charge of c coulombs has entered the skin. Assume that m is invertible and that both m and m^{-1} are differentiable.
- During the treatment, let $q(t)$ be the total electrical charge, in coulombs, that has entered the skin at t minutes after noon. Assume that q is invertible and that both q and q^{-1} are differentiable.

For each of the questions below, circle the one best answer. No points will be given for ambiguous or multiple answers.

a. [2 points] Which of the following expressions represents the total amount of medicine, in mg, that has been absorbed by Zoltan's skin at 12:06 pm?

i. $m(6)$ ii. $\boxed{m(q(6))}$ iii. $m(q(6)+4)$ iv. $m(q(6))+4$
v. $m(6)+4$ vi. $q(m(6))$ vii. $q(m(6)+4)$ viii. $q(m(6))+4$

b. [2 points] Which of the following equations best supports the statement
"Between 12:03 pm and 12:04 pm, Zoltan absorbs about 0.2 mg of the medicine." ?

i. $m(3) = 0.2$ ii. $m(q(4)) = 0.2$ iii. $q'(3) = 0.2$
iv. $m'(q(4)) = 0.2$ v. $m'(3) = 0.2$ vi. $q'(4)\cdot m'(4) = 0.2$
vii. $m'(q'(3)) = 0.2$ viii. $\boxed{q'(4)\cdot m'(q(4)) = 0.2}$ ix. $(q^{-1})'(0.2) = 3$

c. [3 points]
Which of the following is the best interpretation of the equation $\int_0^{30} q'(t)\,dt = 200$?

i. $\boxed{\text{Between noon and 12:30 pm, 200 coulombs of electrical charge enter the skin.}}$

ii. Between noon and 12:30 pm, about 200 coulombs of electrical charge enter the skin.

iii. Between noon and 12:30 pm, electrical charge enters the skin at an average rate of 200 coulombs per minute.

iv. Between noon and 12:30 pm, electrical charge enters the skin at an average rate of about 200 coulombs per minute.

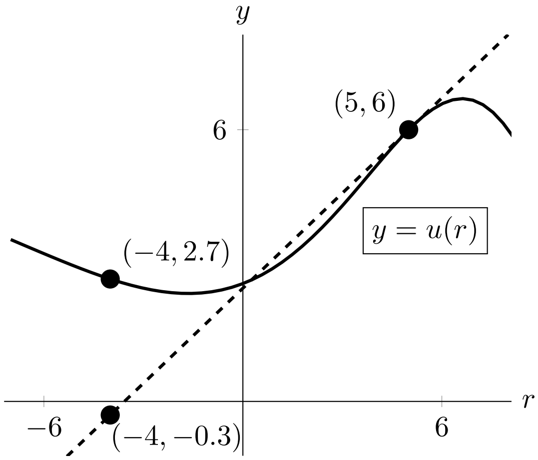
d. [2 points] Which of the following equations expresses the statement:
"Between 12:15 pm and 12:25 pm, Zoltan absorbs an additional 7 mg of the medicine."

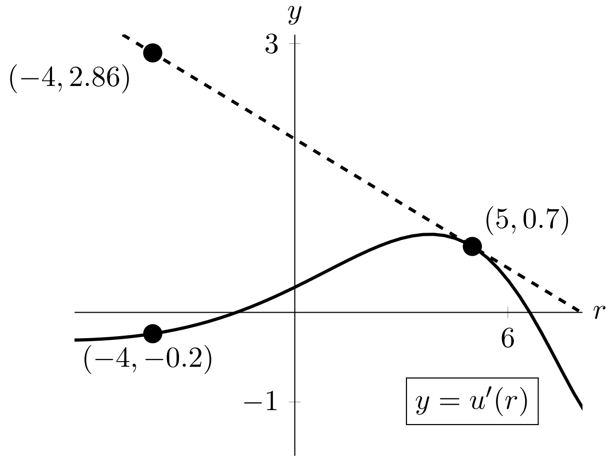
i. $m(25) - m(15) = 7$ ii. $\dfrac{m(25)-m(15)}{10} = 7$ iii. $m'(20) = 0.7$

iv. $\boxed{\int_{q(15)}^{q(25)} m'(c)\,dc = 7}$ v. $\int_{q(15)}^{q(25)} m(c)\,dc = 7$ vi. $\int_{15}^{25} m(c)\,dc = 7$

vii. $\int_{15}^{25} m(q(t))\,dt = 7$ viii. $\int_{15}^{25} m'(q(t))\,dt = 7$

9. [9 points] The graphs of $u(r)$ and $u'(r)$ are shown below. The graphs also show tangent lines to both functions at $r = 5$.





The table below shows some values of $h(s)$ and $h'(s)$. Both h and h' are differentiable.

s	-6	-4	-2	0	2	4	6
$h(s)$	1	4	5	-1	-3	4	7
$h'(s)$	3	2	-4	-1	0	2	1

a. [5 points] Let $g(t) = u(h(t))$. Find a formula for $\ell(t)$, the local linearization of $g(t)$ near $t = -2$, and use this to approximate a solution to $g(t) = 6.14$.

Solution: In order to find $\ell(t)$ we need to find $g(-2)$ and $g'(-2)$.
$g(-2) = u(h(-2)) = u(5) = 6$
$g'(-2) = u'(h(-2))h'(-2) = u'(5) \cdot (-4) = 0.7 \cdot (-4) = -2.8$
So $\ell(t) = g(-2) + g'(-2)(t + 2) = 6 - 2.8(t + 2)$.
To approximate a solution to $g(t) = 6.14$, we want to find a value of t so that $\ell(t) = 6.14$.
Using the formula we found for $\ell(t)$ and solving for t gives us $t = -2.05$.

Answer: $\ell(t) =$ $6 - 2.8(t + 2)$

Answer: $g(t) = 6.14$ when $t \approx$ -2.05

b. [2 points] Write a formula for $c(r)$, the quadratic approximation of $u(r)$ at $r = 5$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: For this we need $u(5) = 6$, $u'(5) = 0.7$,
and $u''(5) = \dfrac{2.86 - 0.7}{-4 - 5} = \dfrac{2.16}{-9} = -0.24$.

Answer: $c(r) =$ $6 + 0.7(r - 5) - 0.12(r - 5)^2$

c. [2 points] Use the data provided to estimate $h''(-5)$.

Answer: $h''(-5) \approx$ $\dfrac{3 - 2}{-6 - (-4)} = \dfrac{-1}{2}$

10. [10 points] Yukiko has a small orchard where she grows Michigan apples. After careful study last season, Yukiko found that the total cost, in dollars, of producing a bushels of apples can be modeled by

$$C(a) = -25500 + 26000e^{0.002a}$$

for $0 \le a \le 320$.

Qabil has promised to buy up to 100 bushels of apples for his famous apple ice cream. If Yukiko has any remaining apples, she has an agreement to sell them to Xanthippe's cider mill at a reduced price. Let $R(a)$ be the revenue generated from selling a bushels of apples. Then

$$R(a) = \begin{cases} 70a & \text{if } 0 \le a \le 100 \\ 2000 + 50a & \text{if } 100 < a \le 320. \end{cases}$$

a. [1 point] How much will Xanthippe's cider mill pay per bushel?

Answer: **\$50**

b. [1 point] What is Yukiko's fixed cost?

Answer: **\$500**

c. [4 points] For what quantities of bushels of apples sold would Yukiko's marginal revenue equal her marginal cost? Write NONE if appropriate.

Solution: Yukiko's marginal revenue is given by

$$MR = \begin{cases} 70 & \text{if } 0 < a < 100 \\ 50 & \text{if } 100 < a < 320 \end{cases}$$

and her marginal cost is $52e^{0.002a}$. We have $52e^{0.002a} = 70$ when $a \approx 148.63$, but this is greater than 100, so it is not in the correct domain. Also $52e^{0.002a} = 50$ when $a \approx -19.61$, which is also not in the domain. Thus there are no values of a where $MC = MR$.

Answer: **None**

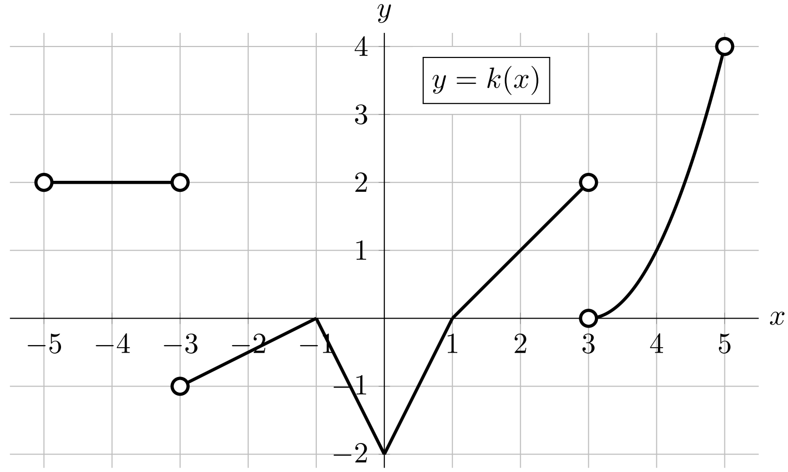
d. [4 points] Assuming Yukiko can produce up to 320 bushels of apples, how many bushels should she produce in order to maximize her profit, and what would that maximum profit be? You must use calculus to find and justify your answer. Make sure to provide enough evidence to justify your answer fully.

Solution: Let $\pi(a) = R(a) - C(a)$ be the profit function. Note that $C(a)$ and $R(a)$ are continuous on this closed interval, so we can apply the Extreme Value Theorem. Since we found in the previous part that MC and MR are never equal, we only need to consider endpoints and points where $\pi'(a)$ does not exist. This happens when $q = 100$. Using the formulas we've been given, we find

$$\pi(0) = -500$$
$$\pi(100) \approx 743.53$$
$$\pi(320) \approx -5808.50$$

Answer: bushels of apples: **100** and max profit: **\$743.53**

11. [10 points] The graph of a portion of $y = k(x)$ is shown below. Note that for $3 < x < 5$, the graph of $k(x)$ is a portion of the graph obtained by shifting $y = x^2$ three units to the right.



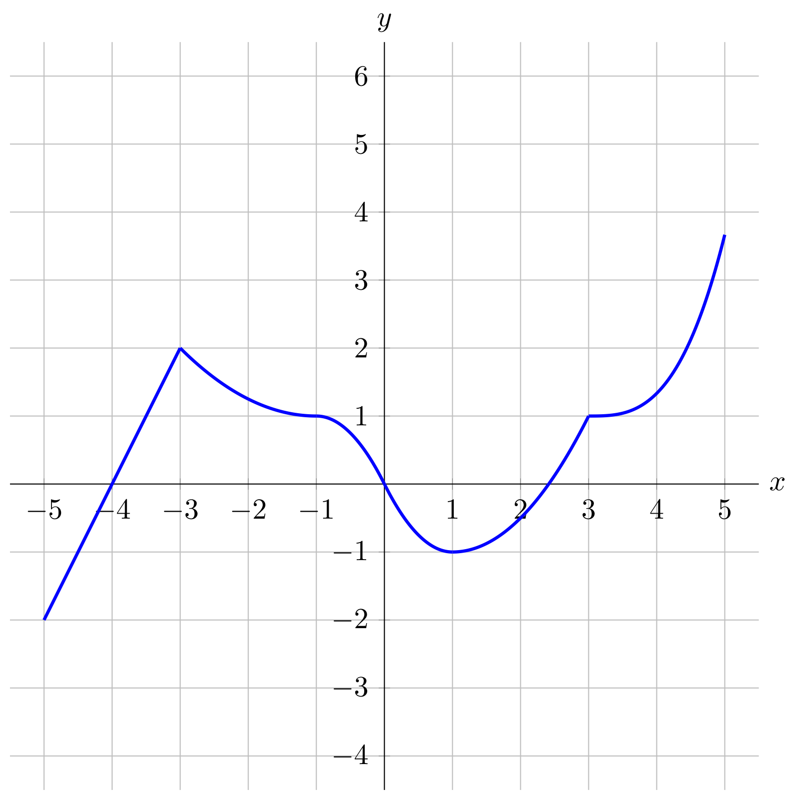
Let $K(x)$ be the continuous antiderivative of $k(x)$ passing through the point $(-1, 1)$.

a. [5 points] Use the graph to complete the table below with the <u>exact</u> values of $K(x)$.

x	-5	-3	-1	1	3	5
$K(x)$	-2	2	1	-1	1	$\frac{11}{3}$

b. [5 points] On the axes below, sketch a detailed graph of $y = K(x)$ for $-5 < x < 5$. Be sure that you pay close attention to each of the following:

- where $K(x)$ is and is not differentiable,
- the values of $K(x)$ you found in the table above,
- where $K(x)$ is increasing/decreasing/constant, and the concavity of $K(x)$.



12. [8 points] Let W be the differentiable function given by

$$W(p) = \begin{cases} 4\ln(2) + 4\ln(-p) & \text{if } p \le -0.5 \\ 2\sin(4p^2 - 1) & \text{if } -0.5 < p < 0.5 \\ \dfrac{\arctan(2p-1)}{p^2} & \text{if } p \ge 0.5. \end{cases}$$

a. [4 points] Use the limit definition of the derivative to write an explicit expression for $W'(3)$. *Your answer should not involve the letter W. Do not evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Answer: $W'(3) = \boxed{\lim_{h\to 0} \dfrac{\frac{\arctan(2\cdot 3 - 1 + h)}{(3+h)^2} - \frac{\arctan(2\cdot 3-1)}{9}}{h}}$

b. [4 points] With W as defined above, consider the function g defined by

$$g(t) = \begin{cases} ct + k & \text{if } t \le 0 \\ W(-e^t) & \text{if } t > 0 \end{cases}$$

for some constants c and k. Find all values of c and k so that $g(t)$ is differentiable.
Show your work carefully, and leave your answers in exact form.
If no such values of c and/or k exist, write NONE in the appropriate answer blank and be sure to justify your reasoning.

Solution: Note that for $t > 0$, $g(t) = 4\ln(2) + 4t$, so

$$g'(t) = \begin{cases} c & \text{for } t < 0 \\ 4 & \text{for } t > 0 \end{cases}$$

(Alternatively, $g'(t) = W'(-e^t) \cdot -e^t$.) Since we are told that W is differentiable, we need only to find values so that g is differentiable at $t = 0$.
In order for g to be differentiable, we need to find values of c and k so that $\lim\limits_{t\to 0^-} g(t) = \lim\limits_{t\to 0^+} g(t)$ and $\lim_{t\to 0^-} g'(t) = \lim_{t\to 0^+} g'(t)$.

The first equation is true when $c \cdot 0 + k = W(-e^0) = W(-1)$. Note that since $-1 < -0.5$, we have $W(-1) = 4\ln(2) + 4\ln(-(-1)) = 4\ln(2)$, so $k = 4\ln(2)$.

The second equation is true when $c = -W'(-1)$. Near $p = -1$ we have $W'(p) = 4$. Therefore, we need $c = 4$.

Answer: $c =$ **4** and $k =$ $4\ln(2)$