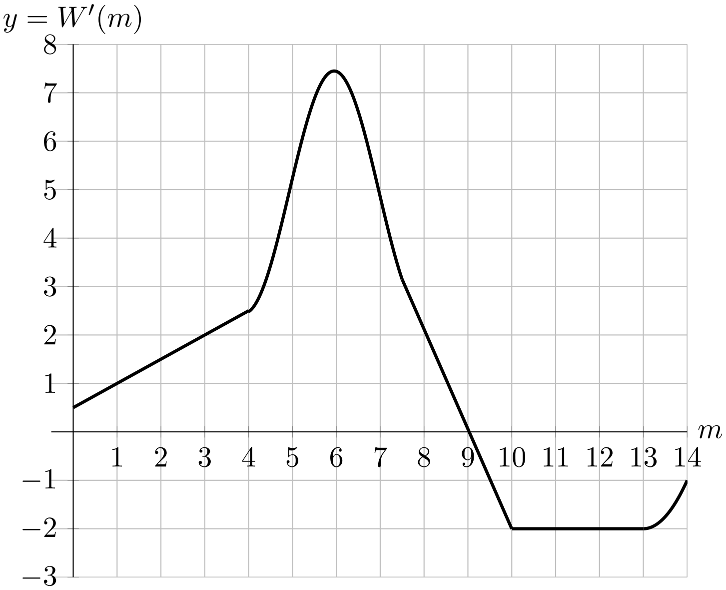
1. [13 points] Let $W(m)$ be the weight, in Newtons, that an ant that is m months old can carry on its back. The graph of $W'(m)$, (*the derivative of* W), is shown below.



Answer the following questions. Write "NI" if there is not enough information to answer the question.

a. [2 points] At what age m, with $0 \le m \le 14$, can an ant carry the most weight on its back?

Solution: $m = 9$.

b. [2 points] At what age m, with $0 \le m \le 14$, is the amount of weight an ant can carry on its back increasing most quickly?

Solution: $m = 6$

c. [2 points] On which, if any, of the following intervals does it appear that the function $W(m)$ is always linear? Circle all correct choices, or circle NONE OF THESE if appropriate.

$(0, 4)$ $\qquad$ $(4, 7)$ $\qquad$ $(8, 10)$ $\qquad$ $\boxed{(10,13)}$ $\qquad$ NONE OF THESE

d. [2 points] On which, if any, of the following intervals does it appear that the function $W(m)$ is always decreasing? Circle all correct choices, or circle NONE OF THESE if appropriate.

$(0, 3)$ $\qquad$ $(6, 9)$ $\qquad$ $\boxed{(9,10)}$ $\qquad$ $\boxed{(10,14)}$ $\qquad$ NONE OF THESE

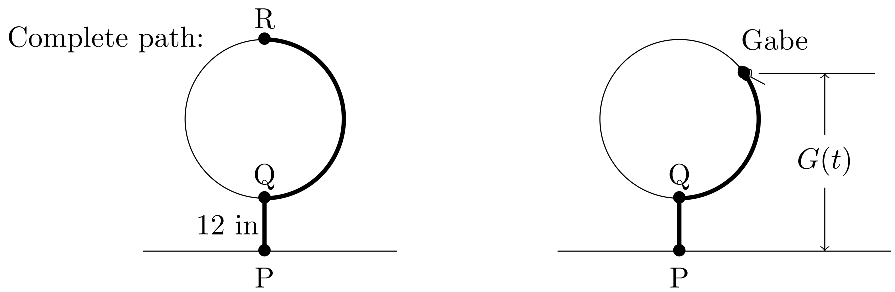
e. [3 points] Complete the following sentence using the fact that $W'(13.5) = -1.75$.

Solution: As the age of an ant increases from 13 months to 13.5 months, the amount of weight it can carry on its back *decreases approximately by* 0.875 *Newtons.*

f. [2 points] In the context of this problem, what are the units of the output values of the function $W'(m)$?

Solution: Newtons per month.

2. [13 points] After Blizzard left Arizona, Gabe the mouse found a large globe (a sphere) to climb. The globe has a diameter of 40 inches and it is attached to a 12-inch-long pole. Gabe starts at the base of the pole at point P. He climbs up to the bottom of the globe at point Q. He then climbs the globe along a semicircle until he stops at the top of the globe at point R (see the diagram below). *Note that the diagram is not drawn to scale.*



a. [8 points] Assume that Gabe walks through the path at a velocity of 3 inches per second. Let $G(t)$ be Gabe's height above the ground (in inches) t seconds after he started his climb at point P. Find a piecewise-defined formula for $G(t)$. *Be sure to include the domain for each piece.*

Solution: From point P to Q: It takes the ant 4 seconds to climb 12 inches at a velocity of 3 inches per second. During that time, the ant climbs at a constant rate of 3 inches per seconds starting at the floor, hence $G(t) = 3t$ for $0 \leq t \leq 4$.

From point Q to R: The distance L along the semicircle traveled by the ant is $L = \frac{1}{2}(2\pi R)$, where R is the radius of the circle. In this case $R = 20$ inches, then $L = 20\pi$. Hence it takes the ant $T = \frac{L}{3} = \frac{20\pi}{3}$ seconds to go from point Q to R a t a velocity of 3 inches per second. Its height is given by a sinusoidal function with midline at $k = 12 + 20 = 32$, amplitude $A = \frac{1}{2}(40) = 20$, period $P = 2T = \frac{40\pi}{3}$ and a minimum at $(4, 12)$. Hence $G(t) = 32 - 20\cos(B(t-4))$. The constant $B = \frac{2\pi}{P} = \frac{2\pi}{\frac{40\pi}{3}} = \frac{3}{20}$ for $4 \leq t \leq 4 + T$. Hence

$$G(t) = \begin{cases} 3t & \text{for} \quad 0 \leq t \leq 4. \\ \\ 32 - 20\cos\left(\frac{3}{20}(t-4)\right) & \text{for} \quad 4 \leq t \leq 4 + \frac{20\pi}{3} \end{cases}$$

b. [5 points] After climbing the globe, Gabe jumps onto a small ferris wheel. Let $H(t)$ be his height, in inches, above the ground t seconds after Gabe jumped, where

$$H(t) = 12 + 9\cos\left(\frac{\pi}{75}(t-120)\right).$$

Find the the *smallest* positive value of t at which Gabe's height above the ground is 10.5 inches. Clearly show each step of your algebraic work. Give your answer in *exact* form.

Solution:

$$12 + 9\cos\left(\frac{\pi}{75}(t-120)\right) = 10.5$$

$$\cos\left(\frac{\pi}{75}(t-120)\right) = -\frac{1}{6}$$

$$\frac{\pi}{75}(t-120) = \cos^{-1}\left(-\frac{1}{6}\right) \qquad t_0 = 120 + \frac{75}{\pi}\cos^{-1}\left(-\frac{1}{6}\right)$$

$$\text{(smallest positive)} \qquad t_{ans} = t_0 - P = \frac{75}{\pi}\cos^{-1}\left(-\frac{1}{6}\right) - 30.$$

where the period of $H(t)$ is $P = \frac{2\pi}{\frac{\pi}{75}} = 150$.

3. [5 points] Let

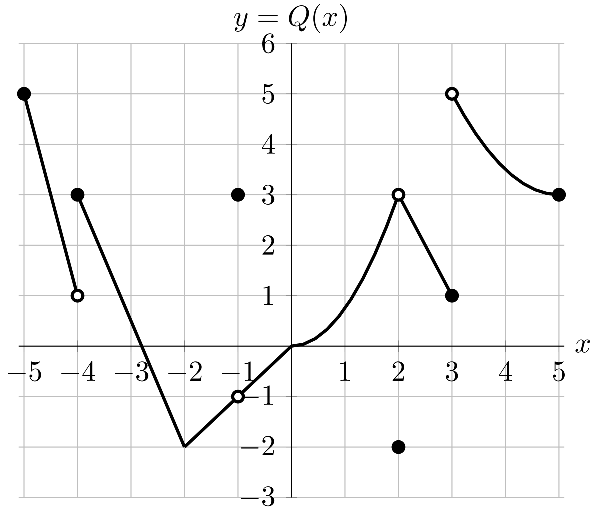
$$B(k) = e^{-4k^2} \tan(k+3).$$

Use the limit definition of the derivative to write an explicit expression for $B'(5)$. *Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Solution:

$$B'(5) = \lim_{h \to 0} \frac{e^{-4(5+h)^2} \tan(h+8) - e^{-100} \tan(8)}{h}.$$

4. [14 points] The graph of a function $Q(x)$ with domain $[-5, 5]$ is shown below.



a. [2 points] On which of the following intervals is $Q(x)$ invertible? Circle all that are true.

$[-4, -1]$ $\qquad$ $[-2, 3]$ $\qquad$ $[2, 5]$ $\qquad$ $[-2, 2]$ $\qquad$ None of these.

b. [8 points] Find the numerical value of the following mathematical expressions. If the answer cannot be determined with the information given, write "NI". If any of the quantities does not exist, write "DNE".

i) Find $\lim_{x \to -1} Q(x)$

Solution:

i) **Answer:** -1.

ii) Find $\lim_{w \to 2} Q(Q(w))$

Solution:

ii) **Answer:** 1.

iii) Find $\lim_{h \to 0} \dfrac{Q(-3+h) - Q(-3)}{h}$

Solution:

iii) **Answer:** -2.5.

iv) Find $\lim_{x \to \infty} Q\left(\dfrac{1}{x} + 3\right)$

Solution:

iv) **Answer:** 5.

v) Find $\lim_{x \to \frac{1}{3}} xQ(3x - 5)$

Solution:

v) **Answer:** DNE.

c. [2 points] For which values of $-5 < x < 5$ is the function $Q(x)$ not continuous?

Solution: $x = -4, -1, 2, 3.$

d. [2 points] For which values of $-5 < p < 5$ is $\lim_{x \to p^-} Q(x) \neq Q(p)$?

Solution: $p = -4, -1, 2.$

5. [9 points] A company is hired to clean the trash accumulated in a lake.

1. Let $T(d)$ be the total amount of recyclable trash collected (in thousands of pounds) after they have cleaned for d days.
2. Let $R(s)$ be the revenue (in thousands of dollars) the company obtains from recycling s thousand pounds of recyclable trash.

Assume that the functions T and R are invertible and differentiable.

a. [4 points] Find a mathematical expression involving the functions T, R, T^{-1} and/or R^{-1} that represents each of the following sentences.

i) The revenue (in thousands of dollars) the company obtains from recycling all the trash collected from this lake if it takes the company 14 days to complete the job.

Solution: $R(T(14))$.

ii) The quantity, in thousands of pounds, of trash collected during the fifth day.

Solution: $T(5) - T(4)$

b. [2 points] Let $H(w)$ be the amount of recyclable trash collected (in *pounds*) during the first w *weeks* after the company started cleaning. Find a formula for $H(w)$ in terms of the functions T and R.

Solution: $H(w) = 1000T(7w)$.

c. [3 points] Circle the <u>one</u> statement below that is best supported by the equation

$$(R^{-1})'(20) = 3.5.$$

i) After the company has recycled enough trash to earn 20,000 dollars in revenue, if they recycle another thousand pounds of trash, then their revenue will be increased by about 3,500 dollars.

ii) Once the company recycles 20,000 pounds, the next thousand pounds of trash recycled will increase their revenue by about 3,500 dollars.

iii) The company earns 20,000 dollars for every 3,500 pounds of trash recycled.

iv) If the company recycles trash until they make 20,000 dollars in revenue, they need to recycle about 3,500 more pounds of trash to make an additional thousand dollars in revenue.

v) After the company has collected 20,000 pounds of recyclable trash, the amount of additional recyclable trash the company would have to collect to increase their revenue by 100 dollars is approximately 350 pounds.

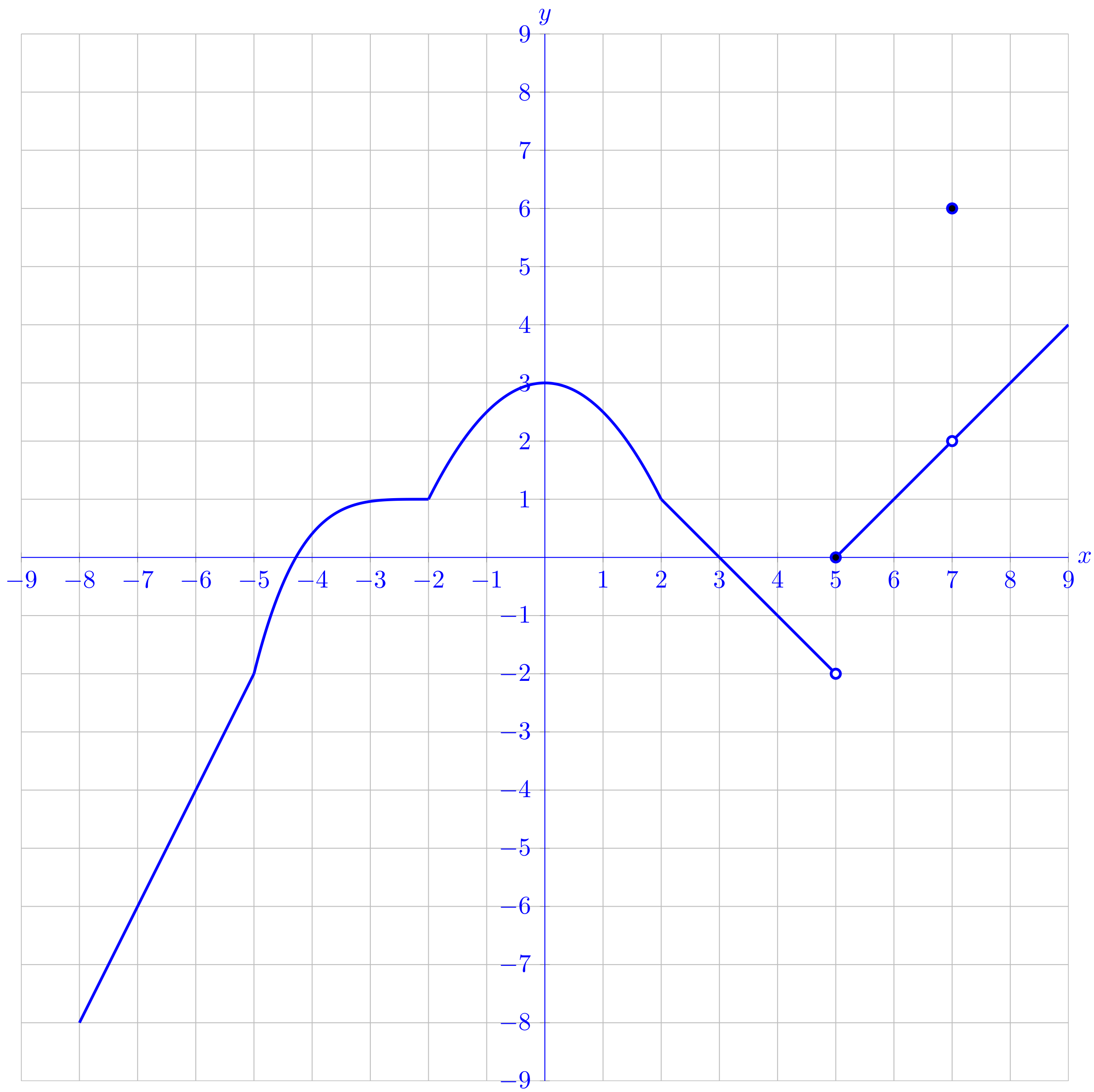
Solution: IV)

6. [9 points] On the axes provided below, sketch the graph of a single function $y = R(x)$ satisfying all of the following conditions:

- The function $R(x)$ is defined on $-8 \le x \le 9$.
- $R'(x) = 2$ for $-8 < x < -5$.
- $R(x)$ is concave down and increasing on $-5 < x < -2$.
- $R(-2) = 1$.
- $R(x) = R(-x)$ for $-2 \le x \le 2$.
- The vertical intercept of $R(x)$ is $y = 3$.
- $\lim_{x\to 5^-} R(x) = -2$ but $\lim_{x\to 5} R(x)$ does not exist.
- $R(x)$ is not continuous at $x = 7$ but $\lim_{x\to 7} R(x)$ exists.

Make sure that your graph is large and unambiguous.

Solution:



7. [13 points] After testing different ingredients in their parents' garages, Imran and Nicole have recently opened new organic peanut butter companies.

a. [3 points] Two months after opening, Imran's company, Chunky Munky, has produced a total of 256 pounds of peanut butter. Imran thinks Chunky Munky produces peanut butter at a constant rate of 690 pounds every 6 months. Assuming Imran is correct, write a formula for $P(m)$, the total amount of peanut butter, in pounds, that Chunky Munky will have produced m months after opening.

Solution: Since the production increases at a constant rate, then $P(m)$ must be a linear function. The slope of $P(m)$ is $\frac{690}{6} = 115$ pounds per month. Since $P(2) = 256$, then using the point slope formula for linear functions we get

$$P(m) = 256 + 115(m - 2).$$

b. [4 points] Nicole's company, Lots O' Crunch, has produced a total of 182 pounds of peanut butter two months after opening and a total 454 pounds of peanut butter five months after opening. Nicole thinks that Lots O' Crunch produces peanut butter exponentially. Assuming Nicole is correct, write a formula for $Q(x)$, the total amount of peanut butter, in pounds, Lots O' Crunch will have produced x months after opening. *Decimal approximations must be rounded to at least three decimal places.*

Solution: We know that $Q(2) = 182$ and $Q(5) = 454$ where $Q(x) = ab^x$. Then

$$ab^5 = 454$$
$$ab^2 = 182$$
$$b^3 = \frac{454}{182}$$
$$b = \left(\frac{454}{182}\right)^{\frac{1}{3}} \approx 1.356 \qquad \text{and} \qquad a = \frac{182}{b^2} = \frac{182}{\left(\frac{454}{182}\right)^{\frac{2}{3}}} \approx 98.95.$$

Then $Q(x) = \frac{182}{\left(\frac{454}{182}\right)^{\frac{2}{3}}} \left(\left(\frac{454}{182}\right)^{\frac{1}{3}}\right)^x \approx 98.95(1.356)^x$.

Ann Arbor's leading local peanut butter company is Sticky PB Company. The total amount of peanut butter produced by Sticky PB Company m months after Chunky Munky opens is given by

$$S(m) = 1500e^{0.32m}.$$

c. [2 points] By what percent is Sticky PB Company's production growing every month? Round your answer to two decimal places.

Solution: Since $b = e^{.32}$, then $r = b - 1 = e^{.32} - 1 \approx 0.38$. Hence it grows by 38% every month.

d. [4 points] After a lot of analysis, Imran determines that Chunky Munky's total peanut butter production m months after opening is best modeled by the exponential function

$$C(m) = 100(1.6)^m.$$

According to this model, when will Chunky Munky and Sticky PB Company have produced the same amount of peanut butter? *Show all your work and leave your answer in exact form.*

Solution:
Method 1:

$$\begin{aligned}
1500e^{0.32m} &= 100(1.6)^m \\
\ln\left(1500e^{0.32m}\right) &= \ln\left(100(1.6)^m\right) \\
\ln(1500) + \ln(e^{.32m}) &= \ln(100) + \ln((1.6)^m) \\
\ln(1500) + 0.32m &= \ln(100) + m\ln(1.6) \\
0.32m - m\ln(1.6) &= \ln(100) - \ln(1500) \\
m(0.32 - \ln(1.6)) &= \ln(100) - \ln(1500) \\
m &= \frac{\ln(100) - \ln(1500)}{0.32 - \ln(1.6)}
\end{aligned}$$

Method 2:

$$1500e^{0.32m} = 100(1.6)^m$$

$$\frac{e^{.32m}}{(1.6)^m} = \frac{1}{15}$$

$$\left(\frac{e^{.32}}{1.6}\right)^m = \frac{1}{15}$$

$$\ln\left(\left(\frac{e^{.32}}{1.6}\right)^m\right) = \ln\left(\frac{1}{15}\right)$$

$$m\ln\left(\frac{e^{.32}}{1.6}\right) = \ln\left(\frac{1}{15}\right) \qquad \text{then} \qquad m = \frac{\ln\left(\frac{1}{15}\right)}{\ln\left(\frac{e^{.32}}{1.6}\right)}$$

8. [9 points] Han is playing with a balloon. He blows it up and then lets it go without tying it and watches it fly straight upwards away from him. Let $B(t)$ be the distance, in inches, of the balloon from Han t seconds after he releases it. You may assume B is invertible on the interval shown below.

t (seconds)	0	0.2	0.6	0.8	0.9	1.2	1.4	1.6
$B(t)$ (inches)	0	0.6	1.0	1.4	1.8	2.4	2.8	3.1

a. [2 points] What is the average velocity of the balloon over the first 0.8 seconds of its flight? Show your work and include units.

Solution: Average velocity$=\dfrac{1.4-0}{0.8-0}=1.75$ inches per second.

b. [2 points] Estimate the instantaneous velocity of the balloon 1.45 seconds after Han releases it. Show your work and include units.

Solution:
Instantaneous velocity of the balloon at $t=1.45 \approx \dfrac{3.1-2.8}{1.6-1.4}=1.5$ inches per second.

c. [3 points] What is the average rate of change of B^{-1} over the interval $[0.6, 1.4]$? Show your work and include units.

Solution:
Average rate of change of B^{-1} over the interval $[0.6, 1.4]$$=\dfrac{0.8-0.2}{1.4-0.6}=\dfrac{3}{4}$ seconds per inch.

d. [2 points] Over which of the following intervals could $B(m)$ be linear? Circle <u>all</u> possible intervals.

Solution:

$0 \le m \le 0.6$ $\quad$ $0.6 \le m \le 0.9$ $\quad$ $\boxed{0.9 \le m \le 1.4}$ $\quad$ $\boxed{1.4 \le m \le 1.6}$ $\quad$ None of these

9. [9 points] Consider the rational function r defined by

$$r(x) = \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)}.$$

For all of the following parts of this problem, leave your answers in *exact* form.

a. [2 points] What is the domain of $r(x)$?

Solution: $x \neq -1, \sqrt{3}$.

b. [2 points] Find the *equations* of all vertical asymptotes of $r(x)$. If there are none, write NONE.

Solution: $x = \sqrt{3}$

c. [2 points] Let $p(x) = 3x^2 + 1.2x - 5$. Find the *equations* of all horizontal asymptotes of $\frac{r(x)}{p(x)}$. If there are none, write NONE. Show your work or reasoning to justify your answer.

Solution: Since $\frac{r(x)}{p(x)} = \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)}$. Then its horizontal asymptote(s) can be found by finding

$$A = \lim_{x\to\infty} \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)} \quad \text{and} \quad B = \lim_{x\to-\infty} \frac{3\left(x-\sqrt{2}\right)(\pi x+7)^2(x+1)}{(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)}.$$

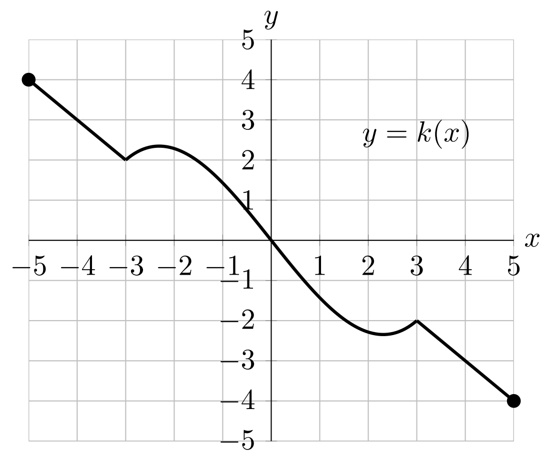
In order to find A and B, we need to notice that the leading terms of $3(x-\sqrt{2})(\pi x+7)^2(x+1)$ and $(x+1)\left(x-\sqrt{3}\right)(3x^2+1.2x-5)$ are $3(x)(\pi x)^2(x) = 3\pi^2x^4$ and $(x)(x)(3x^2) = 3x^4$ respectively. Hence $A = B = \pi^2$. Then the horizontal asymptote of $\frac{r(x)}{p(x)}$ is $y = \pi^2$.

d. [3 points] If $q(x) = \frac{2e^{kx}}{1+2^x}$, find all values of k so that $\lim_{x\to\infty} q(x) = 0$. If there are none, write NONE. Show your work or reasoning to justify your answer.

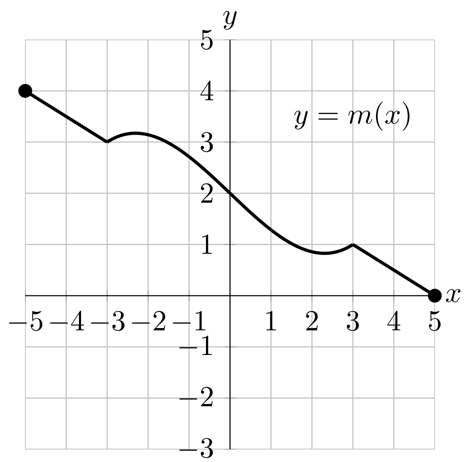
Solution: In order for $\lim_{x\to\infty} q(x) = 0$, the function $y = 2^x$ must dominate $y = e^{kx}$. This is true if the growth factor of $y = 2^x$ is larger than the one of $y = e^{kx}$. Hence we are looking for the values of k such that $2 > e^k$. Hence $k < \ln(2)$.

10. [6 points] A part of the graph of a function $k(x)$ with domain $-5 \le x \le 5$ is given below.

In each of the following parts, the corresponding portion of the graph of a function obtained from k by one or more transformations is shown, together with a list of possible formulas for that function. In each case, circle *all* possible formulas for the function shown. *Note that the graphs are not all drawn at the same scale.*



a. [3 points]



A. $\frac{1}{2}k(x) - 2$

B. $\boxed{\frac{1}{2}k(x) + 2}$

C. $-\frac{1}{2}k(-x) - 2$

D. $\boxed{-\frac{1}{2}k(-x) + 2}$

E. $-\frac{1}{2}k(x) - 2$

F. $2k(x) - 2$

G. $2k(x) + 2$

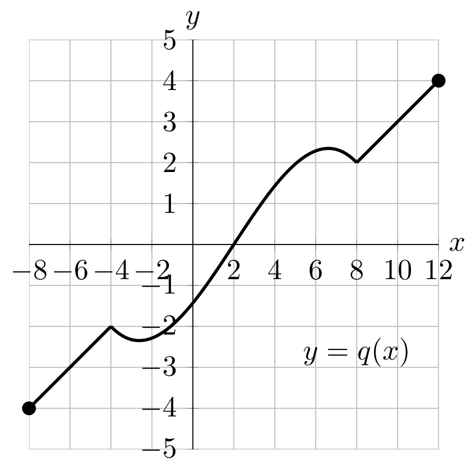
H. $-2k(-x) - 2$

I. $-2k(-x) - 2$

J. $-2k(x) + 2$

K. NONE OF THESE

b. [3 points]



A. $k(2x + 2)$

B. $k(-2x - 2)$

C. $-k(2x + 2)$

D. $k(-2x + 2)$

E. $-k(0.5(x + 2))$

F. $-k(0.5x - 2)$

G. $k(0.5x + 2)$

H. $k(0.5(x - 2))$

I. $k(2(x + 1))$

J. $\boxed{-k(0.5(x - 2))}$

K. NONE OF THESE