Math 115 — Second Midterm — November 13, 2017

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	12	
3	10	
4	15	
5	8	

Problem	Points	Score
6	6	
7	5	
8	11	
9	7	
10	12	
Total	100	

1. [14 points] Let g be a twice differentiable function defined on $-1 < x < 11$. Some values of $g(x)$, $g'(x)$ and $g''(x)$ are shown in the table below.

x	0	2	4	6	8	10
$g(x)$	-2	-1	3	4	5	6
$g'(x)$	0.5	2	?	5	1	2
$g''(x)$	2	1	5	-3	-1	0.5

a. [7 points] Find the *exact* value of the following expressions. If there is not enough information to compute the value, write "NI". Show all your work.

i) Let $h(x) = 2^{g(x)}$. Find $h'(6)$.

Solution: We have that $h'(x) = (\ln 2)2^{g(x)}g'(x)$ and so

$$h'(6) = (\ln 2)2^{g(6)}g'(6) = 80 \ln 2$$

ii) Let $k(x) = g(x)g'(x)$. Find the value of $g'(4)$ if $k'(4) = 15$.

Solution: The product rule gives $k'(x) = g(x)g''(x) + (g'(x))^2$ and so $k'(4) = g(4)g''(4) + (g'(4))^2$. Plugging in values, $15 = 3 \cdot 5 + g'(4)^2$ and so $g'(4) = 0$.

iii) Let $r(x) = \dfrac{\sin(x)}{g(x)}$. Find $r'(0)$.

Solution: By the quotient rule, $r'(x) = \dfrac{g(x)\cos(x) - \sin(x)g'(x)}{g(x)^2}$, and so

$$r'(0) = \frac{g(0)}{g(0)^2} = -1/2.$$

b. [7 points] Let $j(x) = g(14 - 4x)$.

i) Use the values from the table to find a formula for $L(x)$, the linear approximation to $j(x)$ at $x = 2$.

Solution: We have that $j(2) = g(6) = 4$ and $j'(2) = -4g'(6) = -20$. Therefore,

$$L(x) = 4 - 20(x - 2) = -20x + 44.$$

ii) Find an approximate value for $j(2.25)$ using your formula for $L(x)$.

Solution: Using the formula from above,

$$j(2.25) \approx L(2.25) = 4 - 20(0.25) = -1.$$

iii) Is your value an overestimate or underestimate of the exact value of $j(2.25)$? Circle your answer.

OVERESTIMATE   UNDERESTIMATE   NOT ENOUGH INFORMATION

2. [12 points] Let $g(x)$ be a continuous function whose first and second derivatives are given below.

$$g'(x) = e^{2x}(2x-1)^3(x-3)^4 \qquad \text{and} \qquad g''(x) = 4e^{2x}\left(x^2-4\right)(2x-1)^2(x-3)^3$$

a. [6 points] Find all values of x at which $g(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: The critical points of g are when $g'(x) = 0$, so at $x = 1/2$ and $x = 3$. Noticing that $e^{2x} > 0$ for all x, we see that:

- When $x < 1/2$, $g'(x) = (+)(-)(+) = (-)$, so g is decreasing.
- When $1/2 < x < 3$, $g'(x) = (+)(+)(+) = (+)$, so g is increasing.
- When $3 < x$, $g'(x) = (+)(+)(+) = (+)$, so g is increasing.

Therefore, g has a local minimum at $x = 1/2$, and no local maximum.

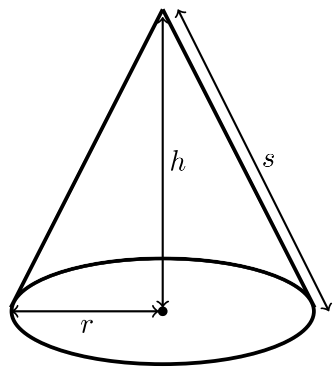
b. [6 points] Find all values of x at which $g(x)$ has an inflection point. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points. Write NONE if $g(x)$ has no points of inflection.

Solution: We see that $g''(x) = 0$ when $x = -2, 2, 1/2, 3$. We still need to check whether g changes concavity at each of these points.

- When $x < -2$, $g''(x) = (+)(+)(+)(-) = (-)$, so g is concave down.
- When $-2 < x < 1/2$, $g''(x) = (+)(-)(+)(-) = (+)$, so g is concave up.
- When $1/2 < x < 2$, $g''(x) = (+)(-)(+)(-) = (+)$, so g is concave up.
- When $2 < x < 3$, $g''(x) = (+)(+)(+)(-) = (-)$, so g is concave down.
- When $3 < x$, $g''(x) = (+)(+)(+)(+) = (+)$, so g is concave up.

Therefore g has inflection points at $x = -2, 2, 3$.

3. [10 points] Jane is designing a water tank using a cone of height h meters and a circular base of radius r meters as shown below.



r = radius
h = height
s = length of slant side

a. [4 points] The cost of the material for the tank is 3 dollars per square meter for the circular base and 5 dollars per square meter for the cone (without the base). The area, A, of the material used for the cone (without the base) is given by the formula $A = \pi r s$ where s is the length of the slant side of the cone, in meters. Find a formula for s in terms of the radius r if Jane plans to spend 200 dollars on the water tank. *Your answer should not include the variable h.*

Solution: The cost of the total tank is equal to the cost of the base + the cost of the cone without the base. So if Jane plans to spend \$200,

$$200 = 3(\pi r^2) + 5(\pi r s)$$

and therefore

$$s = \frac{200 - 3\pi r^2}{5\pi r}.$$

b. [2 points] In the context of this problem, what are appropriate constraints on r and/or s? Choose the <u>one</u> best answer.

$0 < r < \infty$ $\qquad$ $\boxed{0 < r < s}$ $\qquad$ $0 < r < \sqrt{\frac{200}{3\pi}}$ $\qquad$ $0 < s < r$ $\qquad$ $0 < r < \sqrt{\frac{200}{5\pi}}$

c. [4 points] Find a formula for $V(r)$, the volume of the tank (in cubic meters) in terms of the radius r. Recall that the volume of a cone with radius R and height H is $\frac{1}{3}\pi R^2 H$. *Your answer should not include the variables h and/or s.*

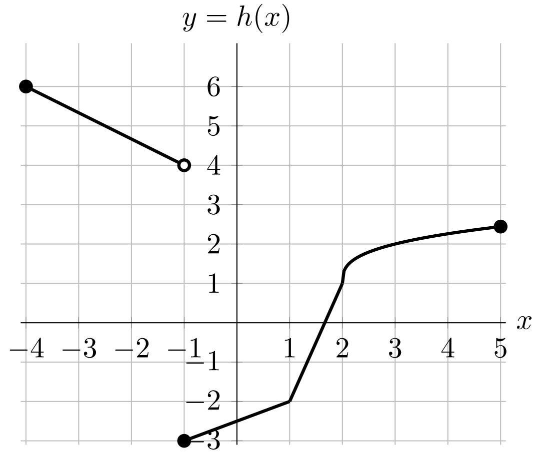
Solution: The volume of the tank is $V = \frac{1}{3}\pi r^2 h$. By the Pythagorean Theorem, $s^2 = r^2 + h^2$, and so $h = \sqrt{s^2 - r^2}$. We also know s in terms of r from the first part of the problem, and so

$$h = \sqrt{\left(\frac{200 - 3\pi r^2}{5\pi r}\right)^2 - r^2}.$$

Therefore,

$$V(r) = \frac{1}{3}\pi r^2 \sqrt{\left(\frac{200 - 3\pi r^2}{5\pi r}\right)^2 - r^2}.$$

4. [15 points] Consider the graph of $h(x)$ below. Note that h is linear on the intervals $[-4,-1)$, $[-1,1]$, and $[1,2]$, differentiable on $(2,5)$, and has a sharp corner at $x=2$.



a. [6 points] Find the exact value of the following expressions. If there is not enough information provided to find the value, write "NI". If the value does not exist, write "DNE". Show all your work.

i) Let $g(x) = xh(x)$. Find $g'(-2)$.

Solution: We have that $g'(x) = h(x) + xh'(x)$. The equation of $h(x)$ on the interval $[-4,1)$ is $h(x) = -2/3x + 10/3$, and so

$$g'(-2) = h(-2) + (-2)h'(-2) = \frac{18}{3} = 6.$$

ii) Let $p(x) = h^{-1}(x)$. Find $p'(0)$.

Solution: We have that $p'(x) = 1/(h'(h^{-1}(x))$. From the graph we see that $h^{-1}(0)$ occurs in the interval $[1,2]$, where $h(x)$ is given by the equation $h(x) = 3x-5$. Therefore, $h^{-1}(0) = 5/3$, so $p'(0) = 1/(h'(5/3))$. Since $1 < 5/3 < 2$, we see that $h'(5/3) = 3$, and so

$$p'(0) = 1/3.$$

b. [2 points] On which of the following intervals does the function $h(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all that apply.

Solution: $[-4,-1]$ $[-2,1]$ $[0,4]$ $\boxed{[2,5]}$ NONE OF THESE

c. [3 points] On which of the following intervals does the function $h(x)$ satisfy the conclusion of the Mean Value Theorem? Circle all that apply.

Solution: $[-4,-1]$ $[-2,1]$ $\boxed{[0,4]}$ $\boxed{[2,5]}$ NONE OF THESE

d. [4 points] For which values given below is the function $m(x) = h(h(x))$ not differentiable? Circle all that apply.

Solution:
$\boxed{x=-3}$ $\boxed{x=-1}$ $\boxed{x=2}$ $\boxed{x=3}$ $x=4$ NONE OF THESE

5. [8 points] Blizzard the snowman and his mouse friend Gabe arrived in Montana, where it has recently snowed. Since Blizzard is still melting, they decide to use this time to pack extra snow onto Blizzard, to help him make it to the North Pole. Let $H(t)$ be Blizzard's height, in inches, if Blizzard and Gabe stay in Montana for t hours. On the interval $1 \le t < \infty$, the function $H(t)$ can be modeled by

$$H(t) = 35 + 10e^{-t/6}(t-2)^{1/3}.$$

Notice that

$$H'(t) = \frac{-5e^{-t/6}(t-4)}{3(t-2)^{2/3}}.$$

a. [6 points] Find all values of t that give global extrema of the function $H(t)$ on the interval $1 \le t < \infty$. Use calculus to find your answers, and be sure to show enough evidence that the point(s) you find are indeed global extrema. For each answer blank, write NONE if appropriate.

Solution: The critical points of $H(t)$ occur at $t = 4$ and $t = 2$. Also checking endpoints, we have that:

- $H(1) = 26.535$
- $H(2) = 35$
- $H(4) = 41.469$
- $\lim_{t\to\infty} H(t) = 35$

and so we see that H has a global maximum when $t = 4$ and a global minimum when $t = 1$.

b. [2 points] Assuming Blizzard stays in Montana for at least 1 hour, what is the tallest height Blizzard can reach? *Remember to include units.*

Solution: Blizzard's tallest height will occur at the global maximum in the interval. Therefore, Blizzard can reach a heigh of 41.469 inches (when he's been in Montana for 4 hours).

6. [6 points] Let $L(x)$ be the linear approximation and $Q(x)$ be the quadratic approximation to the function $d(x)$ near $x = 1$. Suppose that $d'(x)$, $d''(x)$ and $d'''(x)$ are defined for all real numbers. Let $Q(x) = 7(x-1)^2 - 8(x-1) + 3$. Find the *exact* value of the following quantities. If there is not enough information to answer the question, write "NI".

Solution:

$d(0) = \text{NI}$ $\quad d'(1) = -8$ $\quad d''(1) = 14$

$L'(2) = -8$ $\quad Q'''(1) = 0$ $\quad d'''(1) = \text{NI}$

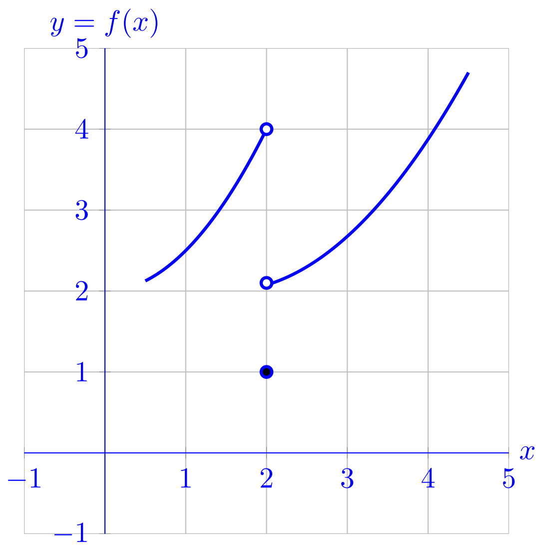
7. [5 points] Sketch graphs of functions $f(x)$ and $g(x)$ satisfying the conditions below, or circle NO SUCH FUNCTION EXISTS. You do not need to explain your answer.

Solution:

A function $f(x)$ defined on the interval $(0, 4)$ that satisfies:

i) $f'(x) > 0$ for all $x \neq 2$.

ii) $x = 2$ is a global minimum.



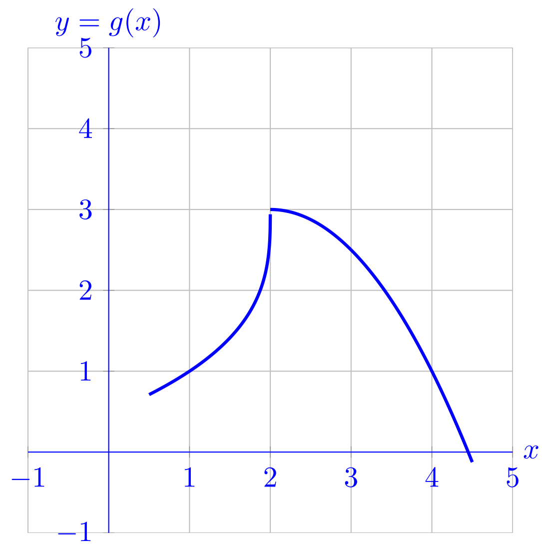
or

NO SUCH FUNCTION EXISTS

A continuous function $g(x)$ defined on the interval $(0, 4)$ that satisfies:

i) $\lim_{x \to 2^-} g'(x) = \infty$.

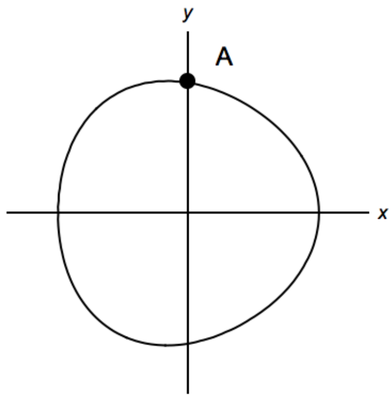
ii) $\lim_{x \to 2^+} g'(x) = 0$.



or

NO SUCH FUNCTION EXISTS

8. [11 points] Let $\mathcal{C}$ be the curve given by the equation $81 - (x^2 + y^2)^2 = 2xy^2$. The graph of $\mathcal{C}$ is shown below.



a. [2 points] Find the coordinates (x, y) of the point A.

Solution: Since the point A lies at the intersection of the y-axis and the curve $\mathcal{C}$, then $x = 0$ and y satisfies $81 - (0^2 + y^2)^2 = 2(0)xy^2$. Hence $y^4 = 81$ or $y = 3$.

$A = (0, 3)$

b. [6 points] Find $\frac{dy}{dx}$. Show all your computations step by step.

Solution:

$$\frac{d}{dx}\left(81 - (x^2 + y^2)^2\right) = \frac{d}{dx}\left(2xy^2\right)$$

$$-2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 2y^2 + 4xy\frac{dy}{dx}$$

$$-4x(x^2 + y^2) - 4y(x^2 + y^2)\frac{dy}{dx} = 2y^2 + 4xy\frac{dy}{dx}$$

$$-4y(x^2 + y^2)\frac{dy}{dx} - 4xy\frac{dy}{dx} = 2y^2 + 4x(x^2 + y^2)$$

$$\frac{dy}{dx} = -\frac{2y^2 + 4x(x^2 + y^2)}{4y(x^2 + y^2) + 4xy}$$

c. [3 points] Find an equation of the tangent line $L(x)$ to the graph of $\mathcal{C}$ at A. Show all your work.

Solution: The slope of $L(x)$ is

$$m = -\frac{2(3)^2 + 4(0)((0)^2 + (3)^2)}{4(3)((0)^2 + (3)^2) + 4(0)(3)} = -\frac{18}{108} = -\frac{1}{6}.$$

Hence using the point A and the slope-intercept formula for the line $L(x)$, we get $L(x) = -\frac{1}{6}x + 3$.

9. [7 points] Let A and B be two constants and

$$h(x) = \begin{cases} 2Bx + A\ln(x) & 0 < x \leq 1 \\ \\ \dfrac{4A}{x} + Bx - 1 & 1 < x \leq 2. \end{cases}$$

Find all the values of A and B that make the function $h(x)$ differentiable on the interval $0 < x < 2$. If no such values exist, write NONE. Justify your answer.

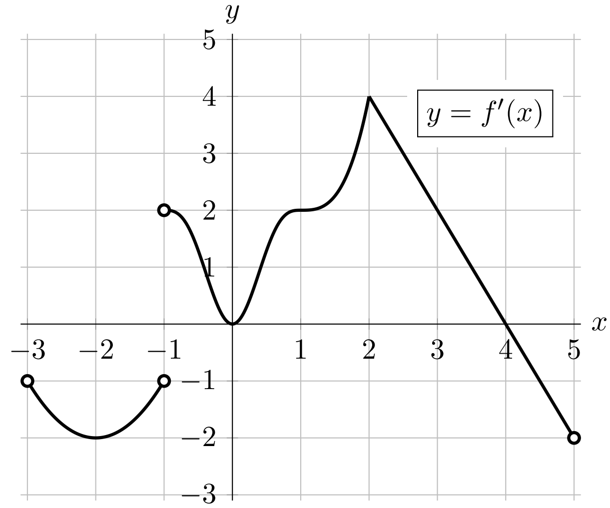
Solution: Since $y = 2Bx + A\ln(x)$ and $y = \frac{4A}{x} + Bx - 1$ are differentiable on the intervals $(0, 1)$ and $(1, 2)$, then we only need to choose A and B so that $h(x)$ is differentiable at $x = 1$.
In order to obtain continuity at $x = 1$, A and B must satisfy

$$\lim_{x\to 1^-} h(x) = \lim_{x\to 1^-} 2Bx + A\ln(x) = \mathbf{2B} = \mathbf{4A} + \mathbf{B} - \mathbf{1} = \lim_{x\to 1^+} \frac{4A}{x} + Bx - 1 = \lim_{x\to 1^+} h(x)$$

This equation can be simplified to $\mathbf{B} = \mathbf{4A} - \mathbf{1}$.
Since $y = 2Bx + A\ln(x)$ and $y = \frac{4A}{x} + Bx - 1$ are differentiable functions on $(0, 4)$, then differentiability of $h(x)$ at $x = 1$ follows if their derivatives $y' = 2B + \frac{A}{x}$ and $y' = -\frac{4A}{x^2} + B$ are equal at $x = 1$. This yields $\mathbf{2B} + \mathbf{A} = -\mathbf{4A} + \mathbf{B}$ or $\mathbf{B} = -\mathbf{5A}$.

Solving both equations $\mathbf{B} = -\mathbf{5A}$ and $\mathbf{B} = \mathbf{4A} - \mathbf{1}$, we get that $-5A = 4A - 1$. Therefore $\mathbf{A} = \dfrac{\mathbf{1}}{\mathbf{9}}$ and then $\mathbf{B} = -\dfrac{\mathbf{5}}{\mathbf{9}}$.

10. [12 points] Let $f(x)$ be a continuous function defined on $-3 < x < 5$. The graph of $f'(x)$ (the <u>derivative</u> of $f(x)$) is shown below. Note that $f'(x)$ has a sharp corner at $x = 2$.



For each of the following parts, circle <u>all</u> of the available correct answers.

a. [2 points] At which of the following values of x does $f(x)$ appear to have a critical point?

Solution:

$x = -2$ [x=-1]   [x=0] $x = 1$ $x = 2$ [x=4]   NONE OF THESE

b. [2 points] At which of the following values of x does $f(x)$ attain a global maximum on the interval $[0, 3]$?

Solution:

$x = 0$ $x = 1$ $x = 2$ [x=3]   NONE OF THESE

c. [2 points] At which of the following values of x does $f(x)$ attain a local minimum?

Solution:

$x = -2$ [x=-1] $x = 0$ $x = 1$ $x = 4$ NONE OF THESE

d. [2 points] Which of the following values of x are <u>not</u> in the domain of $f''(x)$?

Solution:

[x=-1] $x = 0$ $x = 1$ [x=2]   NONE OF THESE

e. [2 points] At which of the following values of x does $f(x)$ appear to have an inflection point?

Solution:

[x=-2]   [x=-1]   [x=0] $x = 1$ $x = 4$ NONE OF THESE

f. [2 points] On which of the following intervals is $f''(x)$ increasing over the <u>entire</u> interval?

Solution:

[(-3,-1)] $(-1, 0)$ $(-1, 1)$ $(0, 2)$ NONE OF THESE