Math 115 — Final Exam — December 14, 2017

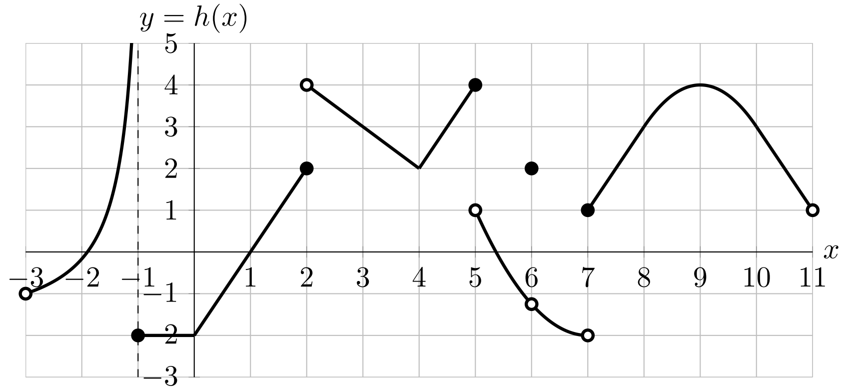
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	10	
3	8	
4	10	
5	12	

Problem	Points	Score
6	12	
7	7	
8	11	
9	9	
10	8	
Total	100	

1. [13 points] The graph of a portion of a function $y = h(x)$ is shown below. Note that the graph is linear where it appears to be linear, including on the intervals $[7, 8]$ and $[10, 11)$.



a. [2 points] At which of the following points p is $h(x)$ *not* continuous at $x = p$? Circle *all* such values.

Solution: [p=-1] $p = 1$ [p=2] $p = 4$ [p=5] NONE OF THESE

b. [2 points] For which of the following values a is $\lim_{x \to a^+} h(x) = h(a)$? Circle *all* such values.

Solution: [a=-1] $a = 2$ [a=4] $a = 5$ $a = 6$ NONE OF THESE

For parts **c.**–**e.**, find the exact value of each of the expressions. If the value does not exist, write DNE. If there is not enough information, write NI.

c. [2 points] Calculate the average value of $h(x)$ on the interval $[-1, 1]$.

Solution:

$$\frac{1}{1-(-1)} \int_{-1}^{1} h(x)dx = \frac{1}{2} \int_{-1}^{1} h(x)dx = \frac{1}{2}(-3) = -1.5.$$

Answer= -1.5.

d. [4 points] Suppose $g(x) = h(3h(x))$. Calculate $g'(1.5)$. Show all your computations to receive full credit.

Solution:

$$g'(x) = h'(3h(x))(3h(x))' = 3h'(3h(x))h'(x).$$

Then $g'(1.5) = 3h'(3h(1.5))h'(1.5) = 3h'(3(1))(2) = 6h'(3) = 6(-1) = -6$

Answer= -6.

e. [3 points] Calculate $\int_{7.5}^{10.5} h''(x)\, dx$.

Solution: Using the Fundamental Theorem of Calculus we obtain

$$\int_{7.5}^{10.5} h''(x)\, dx = h'(10.5) - h'(7.5) = (-2) - (2) = -4.$$

Answer= -4.

2. [10 points] Jane has a company that produces a protein powder for an energy shake. The cost, in dollars, of producing m pounds of protein powder is given by the function

$$C(m) = \begin{cases} \frac{1}{4}(m+2)^2 + 8 & 0 \le m < 16 \\ 2m + 57 & 16 \le m \le 30. \end{cases}$$

The revenue, in dollars, of selling m pounds of protein powder is given by $R(m) = 5m$.

a. [1 point] What is the price, in dollars, at which Jane sells each pound of the protein powder?

Solution: **Answer=** 5

b. [1 point] What is the fixed cost, in dollars, of producing Jane's protein powder?

Solution: Fixed cost is $C(0) = \frac{1}{4}(0+2)^2 + 8 = 9$. **Answer=** 9

c. [2 points] Find all values of $16 \le m \le 30$ for which Jane's profit is positive.

Solution: Jane breaks even when $R(m) = C(m)$. That occurs on $16 \le m \le 30$ when $2m + 57 = 5m$. Then Jane breaks even if $m = 19$. We can see that the profit of selling 20 pounds ($m = 20$) is $5(20) - (2(20) + 57) = 3 > 0$. Since both $R(m)$ and $C(m)$ are continuous on $[16, 30]$ then Jane's profit is positive for $19 < m \le 30$.

Answer: $19 < m \le 30$

d. [2 points] Find all the values of $0 \le m \le 30$ where the marginal cost is equal to the marginal revenue for the protein powder. Show all your work to justify your answer.

Solution: Note that

$$MC(m) = \begin{cases} \frac{1}{2}(m+2) & 0 < m < 16 \\ 2 & 16 < m < 30. \end{cases}$$

and $MR(m) = 5$. Hence $MC = MR$ if $\frac{1}{2}(m+2) = 5$ on $0 < m < 16$. Solving for m we get $m = 8$. We do not consider the interval $16 < m < 30$ since in this case, there is no m that yields $MC = MR$.

Answer= 8

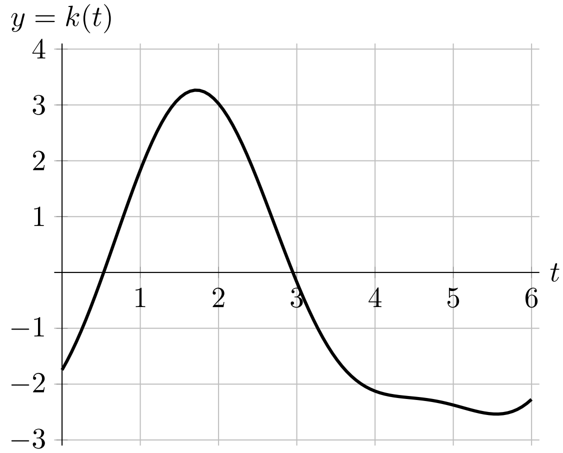
e. [4 points] What is the maximum profit that Jane can make if she sells at most 30 pounds of protein powder? Use calculus to find and justify your answer, and make sure to provide enough evidence to fully justify your answer.

Solution: To find the global maximum of the profit $P(m)$ in dollars of selling m pounds of protein powder, we first need to find its critical points on $0 \le m \le 30$. Critical points satisfy either $MC = MR$ ($P'(m) = 0$) or $P'(m)$ does not exist. Hence $m = 8$ is a critical point. The value $m = 16$ is a critical point of $P(m) = R(m) - C(m)$ since $C(m)$ is not differentiable at $m = 16$.

m	0	8	16	30
$P(m) = R(m) - C(m)$	-5	7	-9	33

Answer: 33 dollars.

3. [8 points] A group of biologists is studying the population of trout in a lake. Let $k(t)$ be the rate at which the population of trout changes, in thousands of trout per month, t months after the biologists started their study, and let $P(t)$ be the population of trout, in thousands, t months after the study begins. The graph of $y = k(t)$ is shown below for $0 \le t \le 6$.



a. [4 points] Fill in the numbers I. - V. in the blanks below to list the quantities in order from least to greatest.

I. The number zero.

II. $P(4) - P(1)$

III. $\int_1^3 k(t)\, dt$

IV. $\int_3^5 k(t)\, dt$

V. $\int_3^5 k(5)\, dt$

Solution: $V \le IV \le I \le II \le III$

b. [3 points] Suppose $P(2) = 8.6$. Use the graph to find a formula for $L(t)$, the linear approximation for $P(t)$ near $t = 2$.

Solution: Since $k(t) = P'(t)$, then $L(t) = P(2) + k(2)(t - 2) = 8.6 + 3(t - 2)$

$L(t) = 8.6 + 3(t - 2)$

c. [1 point] Use $L(t)$ to approximate the population of trout, in thousands, 1.75 months after the study starts.

Solution: $L(1.75) = 8.6 + 3(1.75 - 2) = 8.6 - 0.75 = 7.85.$

$P(1.75) \approx 7.85$

4. [10 points] Gabe the mouse is swimming alone in a very large puddle of water. He keeps track of his swimming time by logging his velocity at various points in time. Gabe starts at a point on the edge of the puddle and swims in a straight line with increasing speed. A table of Gabe's velocity $V(t)$, in feet per second, t seconds after he begins swimming is given below.

t	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$V(t)$	0	0.3	0.4	0.45	0.9	1.2	1.8	2.4	2.7	2.9	3	3.2	3.5

a. [3 points] Give a practical interpretation of the integral $\int_1^{5.5} V(t)\, dt$ in the context of the problem. Be sure to include units.

Solution: The distance Gabe traveled, in feet, in between seconds 1 and 5.5 after he started swimming.

b. [3 points] Estimate $\int_1^{5.5} V(t)\, dt$ by using a right-hand Riemann sum with 3 equal subdivisions. Make sure to write down all terms in your sum.

Solution: If we divide the interval $[1, 5.5]$ in three, we obtain $\Delta t = \dfrac{5.5-1}{3} = 1.5$. Then

$$Right(3) = (V(2.5) + V(4) + V(5.5))\Delta t = (1.2 + 2.7 + 3.2)(1.5) = (7.1)(1.5) = 10.65.$$

Answer=10.65 feet.

c. [1 point] Is your estimate from above an overestimate or an underestimate of the exact value of $\int_1^{5.5} V(t)\, dt$? Circle your answer.

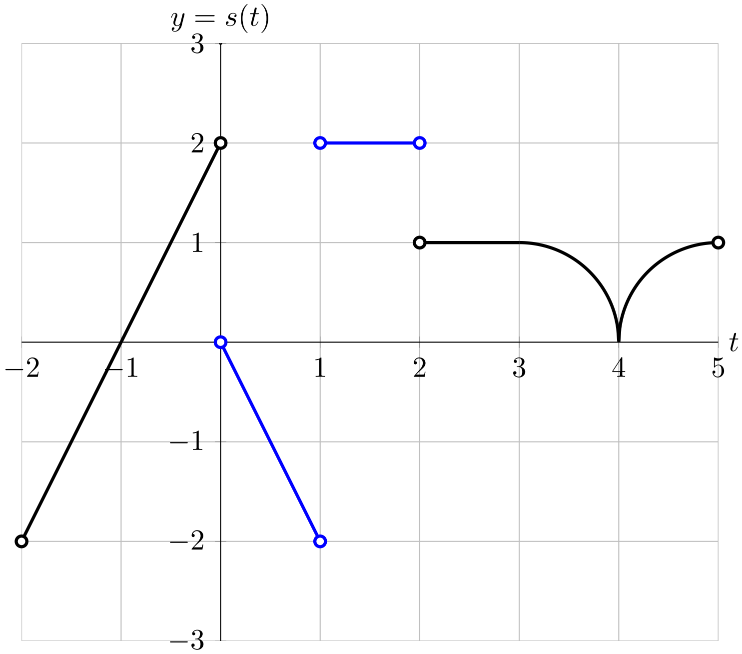
Solution: [OVERESTIMATE] UNDERESTIMATE NOT ENOUGH INFORMATION

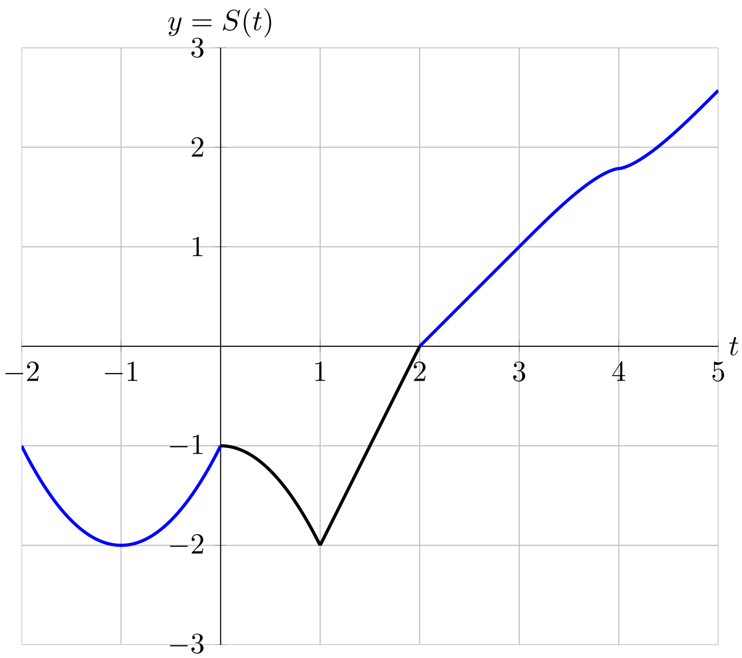
d. [3 points] Suppose Gabe wants to use a Riemann sum to calculate how far he traveled between $t = 1$ and $t = 5.5$, accurate to within 0.15 feet. How many times would he have to measure his velocity in this interval in order to achieve this accuracy? Justify your answer.

Solution: Since $V(t)$ is increasing in $[1, 5.5]$ then $|V(5.5) - V(1)|\Delta t \leq 0.15$, where Δt is the possible width of each interval in order for the estimate to be true. Hence $\Delta t \leq \frac{0.15}{2.8}$. Then if N is the number of times Gabe has to measure his velocity to attain its desired accuracy, then $N = \dfrac{5.5-1}{\Delta t} \geq \dfrac{4.5}{\frac{0.15}{2.8}} = 3(28) = 84$

Answer= At least 84 times.

5. [12 points] A portion of the graphs of two functions $y = s(t)$ and $y = S(t)$ are shown below. Suppose that $S(t)$ is the continuous antiderivative of $s(t)$ passing through the point $(0, -1)$. Note that the graphs are linear anywhere they appear to be linear, and that on the intervals $(3, 4)$ and $(4, 5)$, the graph of $s(t)$ is a quarter circle.





a. [4 points] Use the portions of the graphs to fill in the *exact* values of $S(t)$ in the table below.

t	$S(t)$
-2	-1
-1	-2
0	-1
2	0
3	1
5	$1 + \pi/2$

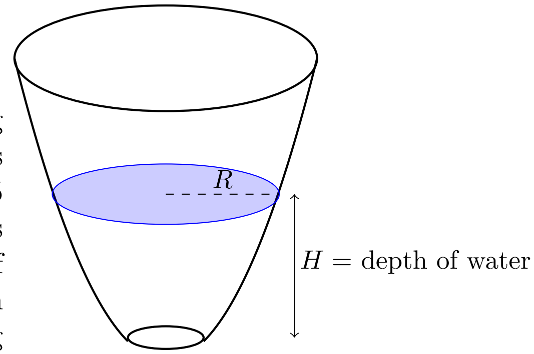
b. [8 points] On the axes above, sketch the missing portions of *both* s and S over the interval $-2 < t < 5$. Make sure to pay attention to:

- the values of $S(t)$ from the table above
- where S is and is not differentiable
- where S and s are increasing/decreasing/constant
- the concavity of the graph $y = S(t)$.

6. [12 points] Water is being poured into a large vase with a circular base. Let $V(t)$ be the volume of water in the vase, in cubic inches, t minutes after the water started being poured into the vase. Let H be the depth of the water in the vase, in inches, and let R be the radius of the surface of the water, in inches.

A formula for V in terms of R and H is given by

$$V = \frac{1}{2}\pi H(R^2 + 8).$$



a. [6 points] Suppose that the water is being poured into the vase at rate of 300 cubic inches per minute. When the depth of the water is 5 inches, the radius of the surface of the water is 4 inches and the radius is increasing at a rate of 1.2 inches per minute. Find the rate at which the depth of the water in the vase is increasing at that time. Show all your work *carefully*.

Solution: Differentiating with respect to time

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{2}\pi H(R^2+8)\right)$$

$$\frac{dV}{dt} = \frac{1}{2}\pi\left(\frac{dH}{dt}(R^2+8) + H\frac{d}{dt}(R^2+8)\right)$$

$$\frac{dV}{dt} = \frac{1}{2}\pi\left(\frac{dH}{dt}(R^2+8) + 2HR\frac{dR}{dt}\right)$$

$$300 = \frac{1}{2}\pi\left(\frac{dH}{dt}((4)^2+8) + 2(5)(4)(1.2)\right) \qquad 300 = \frac{1}{2}\pi\left(24\frac{dH}{dt} + 48\right)$$

$$\frac{dH}{dt} = \frac{\frac{600}{\pi} - 48}{24} \approx 5.96.$$

b. [2 points] Estimate the instantaneous rate of change of H when $t = 3$ if

t	1.5	2.3	3.0	3.2
H	1.3	1.7	1.9	1.95

Show your work and include units.

Solution: $H'(3) \approx \dfrac{1.95 - 1.9}{3.2 - 3} = \dfrac{0.05}{0.2} = 0.25$ inches per minute.

c. [4 points] Recall that R gives the radius of the surface of the water, in inches, t minutes after the water started being poured into the vase. Suppose that R is given by $R = m(t)$ and $m'(3) = 0.7$. Use these facts to complete the following sentence:

Solution: After the water has been poured into the vase for three minutes, over the next ten seconds, the radius of the surface of the water increases approximately by $\frac{7}{60}$ inches.

7. [7 points] Let A and B be positive constants and $f(x) = \dfrac{A(x^2 - B)}{\sqrt{x-3}}$, for $x > 3$. Note that

$$f'(x) = \frac{A(3x^2 - 12x + B)}{2(x-3)^{\frac{3}{2}}} \qquad \text{and} \qquad f''(x) = \frac{3A(x^2 - 8x + 24 - B)}{4(x-3)^{\frac{5}{2}}}.$$

Find all values of A and B so that $f(x)$ has an inflection point at $(8, 2)$. Use calculus to justify that the point $(8, 2)$ is an inflection point. If there are no such values, write NONE.

Solution: In order to have an inflection point at $(8, 2)$ three conditions need to be satisfied:

- $f''(8) = 0$. This is equivalent to

$$0 = \frac{3A((8)^2 - 8(8) + 24 - B)}{4(8-3)^{\frac{5}{2}}}.$$

  This is true if $B = 24$ or $A = 0$. Only $B = 24$ is possible since $A > 0$.

- $f(8) = 2$. This yields

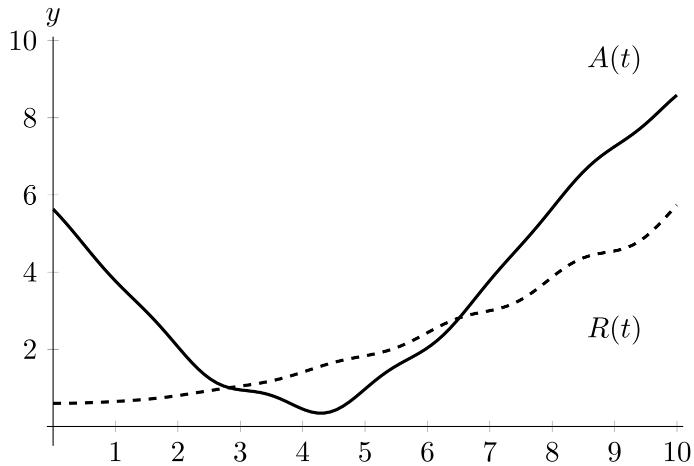
$$2 = \frac{A((8)^2 - B)}{\sqrt{8-3}} = \frac{A(64 - B)}{\sqrt{5}}.$$

  Using $B = 24$ we get $A = \dfrac{\sqrt{5}}{20}$.

- $f''(x)$ needs to have different signs on $(3, 8)$ and $(8, \infty)$.
  - On $(3, 8)$ plug $x = 4$ into $f''(x) = \dfrac{3\sqrt{5}x(x-8)}{80(x-3)^{\frac{5}{2}}}$. We get $f''(4) = \frac{(+)(-)}{+} = -$.
  - On $(8, \infty)$, plug $x = 9$ into $f''(x) = \dfrac{3\sqrt{5}x(x-8)}{80(x-3)^{\frac{5}{2}}}$.We get $f''(9) = \frac{(+)(+)}{+} = +$.

$$A = \frac{\sqrt{5}}{20} \qquad B = 24$$

8. [11 points] A tank contains 30 gallons of water. Beginning at 11 am, water is pumped in and out of the tank. Let $A(t)$ be the rate, in gallons per minute, at which the water is added into the tank t minutes after 11 am. Similarly, let $R(t)$ be the rate, in gallons per minute, at which the water is removed from the tank t minutes after 11 am. The graphs of the functions $A(t)$ (solid line) and $R(t)$ (dashed line) for $0 \le t \le 10$ are shown below.



a. [2 points] For which values of t is the total amount of water in the tank decreasing? Estimate your answer.

Solution: **Answer:** Approximately for $2.75 \le t \le 6.5$.

b. [1 point] At what time $0 \le t \le 10$ does the tank have the least amount of water?

Solution: **Answer:** $t = 0$.

In parts **c**. and **d**., give a mathematical expression that may involve $A(t)$, $R(t)$, their derivatives, and/or definite integrals.

c. [2 points] Find an expression for the total amount of water, in gallons, that was removed from the tank between 11:02 am and 11:05 am.

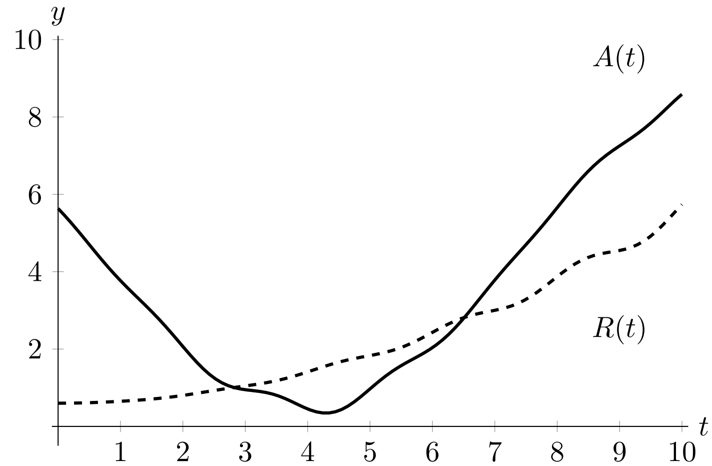
Solution: **Answer:** $\int_2^5 R(t)dt$.

d. [4 points] Find an expression for the amount of water, in gallons, in the tank at 11:10 am.

Solution: **Answer:** $30 + \int_0^{10} A(t) - R(t)dt$.

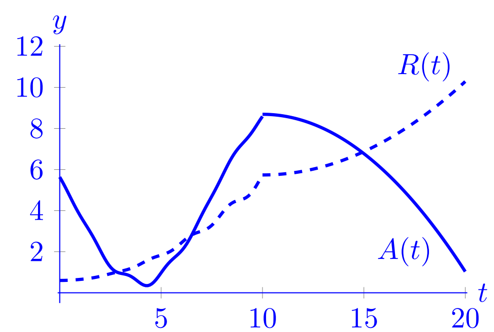
Problem continues on the next page

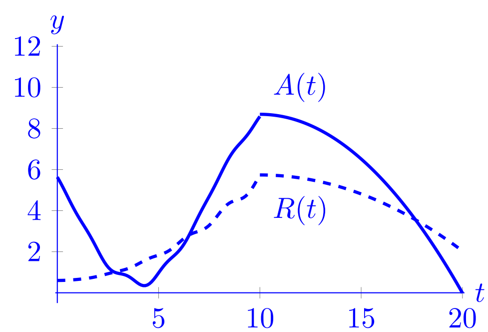
For your convenience, the graphs of $A(t)$ and $R(t)$ for $0 \leq t \leq 10$ are reprinted below.

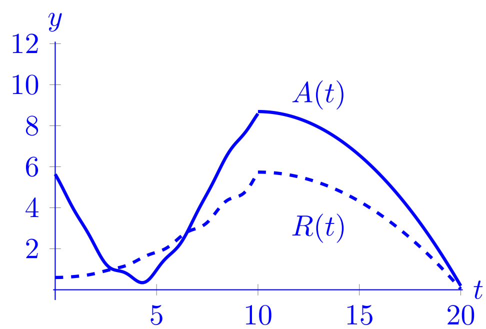


e. [2 points] Suppose that there are 30 gallons of water in the tank at 11:20 am. Which of the following graphs could be the graph of $A(t)$ and $R(t)$ for $0 \leq t \leq 20$ in this case? Circle the *one* best answer.

Solution:







9. [9 points] For the following problems, choose the correct answer. If none of the choices are correct, circle NONE OF THESE.

a. [2 points] Which of the following is an antiderivative of the function $1/x + \cos(x)$ for $x > 0$? Circle *all* correct answers.

Solution:

i. $-\dfrac{1}{x^2} - \sin(x)$

ii. $\boxed{\ln(5x) + \sin(x)}$

iii. $\boxed{\ln(x) + \sin(x) - 20}$

iv. $\ln\left(\dfrac{1}{x}\cos(x)\right)$

v. $\dfrac{1}{x^2} + \sin(x)$

vi. NONE OF THESE

b. [2 points] Suppose $f(x)$ is a differentiable, invertible function defined on $(-\infty, \infty)$ with $f'(x) > 0$ for all x. Suppose that $f(3) = 5$ and $f'(3) = 2$. Which of the following statements must be true? Circle *all* correct answers.

Solution:

i. $\boxed{f'(f^{-1}(x)) = \dfrac{1}{(f^{-1})'(x)}}$

ii. $f'(x)$ is invertible

iii. $(f^{-1})'(x) = \dfrac{1}{f'(x)}$

iv. $\boxed{(f^{-1})'(5) = \dfrac{1}{2}}$

v. $f'(2) = \dfrac{1}{5}$

vi. NONE OF THESE

c. [2 points] If $p(t)$ is an even function that is differentiable on $(-\infty, \infty)$, which of the following must be true? Circle *all* correct answers.

Solution:

i. $\boxed{\displaystyle\int_1^4 p(t)\,dt = \int_{-4}^{-1} p(t)\,dt}$

ii. $\int_{-4}^{4} p(t)\,dt = 0.$

iii. Any antiderivative of $p(t)$ is an even function

iv. $\int_6^8 p(t+3)\,dt = \int_3^5 p(t)\,dt$

v. $\boxed{\displaystyle\int_{-5}^{5} p'(t)\,dt = 0}$

vi. NONE OF THESE

d. [3 points] Suppose the limit definition of the derivative gives

$$g'(-1) = \lim_{h\to 0} \frac{2^{c(-1+h)} + a(-1+h)^3 - (2^{-c} - a)}{h},$$

where a and c are nonzero constants. Which of the following could be the formula for $g(x)$? Circle the *one* best answer.

Solution:

i. $g(x) = 2^{-cx} + ax$

ii. $g(x) = a(x-1)^3 + c^x$

iii. $g(x) = 2^c - a$

iv. $g(x) = 2^{c(x+h)} + ah^3$

v. $\boxed{g(x) = 2^{cx} + ax^3}$

vi. NONE OF THESE

10. [8 points] Consider the family of functions $g(x) = e^x - kx$, where k is a positive constant.

a. [2 points] Show that the point $\Big(\ln(k)\ ,\ k - k\ln(k)\Big)$ is the only critical point of $g(x)$ for all positive k. Show all your work to receive full credit.

Solution: $g'(x) = e^x - k$, then $g'(x) = 0$ if $e^x = k$ or $x = \ln(k)$. There are no other critical points since $g'(x)$ is defined for all values of x.
The y-coordinate of the critical points is given by $g(\ln(k)) = e^{\ln(k)} - k\ln(k) = k - k\ln(k)$.

b. [2 points] Show that $g(x)$ has a global minimum on $(-\infty, \infty)$ at $x = \ln(k)$. Use calculus to justify your answer.

Solution: Since $e^x - kx \to \infty$ as $x \to \infty$ and $e^x - kx \to \infty$ as $x \to -\infty$ and $x = \ln(k)$ is the only critical point, then it is a global minimum.

c. [4 points] Find all values of $0.5 \leq k \leq 2$ that maximize the y-value of the global minimum of $g(x)$ on $(-\infty, \infty)$. Use calculus to justify your answer. Write NONE if no such value exists.

Solution: Let $h(k) = k - k\ln(k)$ defined on $0.5 \leq k \leq 2$. To find critical points, we find where $h'(k) = -\ln(k) = 0$. This occurs at $k = 1$. Looking at the table of values

k	0.5	1	2
$h(k)$	$0.5(1 - \ln(0.5)) \approx 0.84$	1	$2(1 - \ln(2)) \approx 0.61$

Then the value of k that maximizes the value of the global minimum of $g(x)$ is $k = 1$.

Answer: $k = 1$.