Math 115 — First Midterm — October 8, 2018

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	12	
3	6	
4	14	
5	16	

Problem	Points	Score
6	5	
7	9	
8	10	
9	11	
10	6	
Total	100	

1. [11 points] Brianna rides her unicycle north from her home to the grocery store and back again. The differentiable function $r(t)$ represents Brianna's distance in meters from her home t minutes after she leaves the house. Some values of $r(t)$ are shown in the table below.

t	0	1	5	7	10	12	14	16	17
$r(t)$	0	180	1050	1420	1425	980	570	220	0

a. [2 points] What was Brianna's average velocity between times $t = 7$ and $t = 12$? Include units.

Solution: Average velocity$=\dfrac{980-1420}{12-7} = \dfrac{}{5} = -88$ **Answer:** -88 meters per minute.

b. [2 points] Approximate the value of $r'(14)$. Include units.

Solution: $r'(14) \approx \dfrac{220-570}{2} = -175$ **Answer:** -175 meters per minute.

c. [3 points] For which of the following time interval(s) is it possible for $r(t)$ to be concave up on the entire interval? Circle all correct choices.

Solution: Computing average rate of changes in consecutive subintervals we see that

Intervals	[1,5]	[5,7]	[10,12]	[12,14]
Average rate of change	$\frac{870}{4} = 217.5$	$\frac{370}{2} = 185$	$-\frac{445}{2} = -222.5$	$-\frac{410}{2} = -205$

Since the average rate of change only increases on $[10, 14]$, then it is possible that $r(t)$ is concave up on $[\mathbf{10}, \mathbf{14}]$.

Use the following additional information about Brianna's ride to answer the questions below:

- The grocery store is 1430 meters away from Brianna's home.
- It takes Brianna 8 minutes to get to the store.
- On her way to the store, Brianna does not stop at all. On her way back, she only stops once at a traffic light, which is 250 meters from her home.

d. [2 points] For which of the following time interval(s) is $r'(t)$ equal to 0 for some value of t in that interval? Circle all correct choices.

Solution: Based on the information given $r'(t) \neq 0$ on $[1, 5]$ and $[10, 12]$. $r'(8) = 0$ since it takes 8 minutes to get to the store. Since she stops on her way back, then $r'(t) = 0$ for $14 \leq t \leq 16$.

[1,5] $\boxed{[5,10]}$ [10,12] $\boxed{[12,16]}$ NONE OF THESE

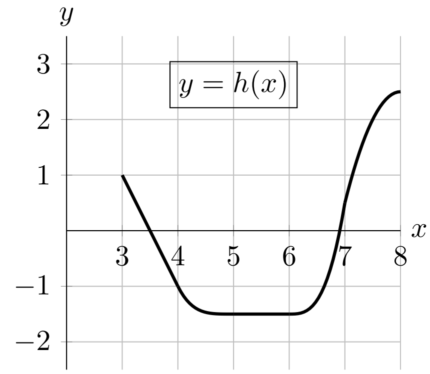
e. [2 points] For which of the following time interval(s) is $r'(t)$ negative for some value of t in that interval? Circle all correct choices.

Solution: The derivative of $r(t)$ is negative on her way back.

[1,5] $\boxed{[5,10]}$ $\boxed{[10,12]}$ $\boxed{[12,16]}$ NONE OF THESE

2. [12 points] On the axes provided below, sketch the graph of a single function $y = Q(x)$ satisfying all of the following conditions:

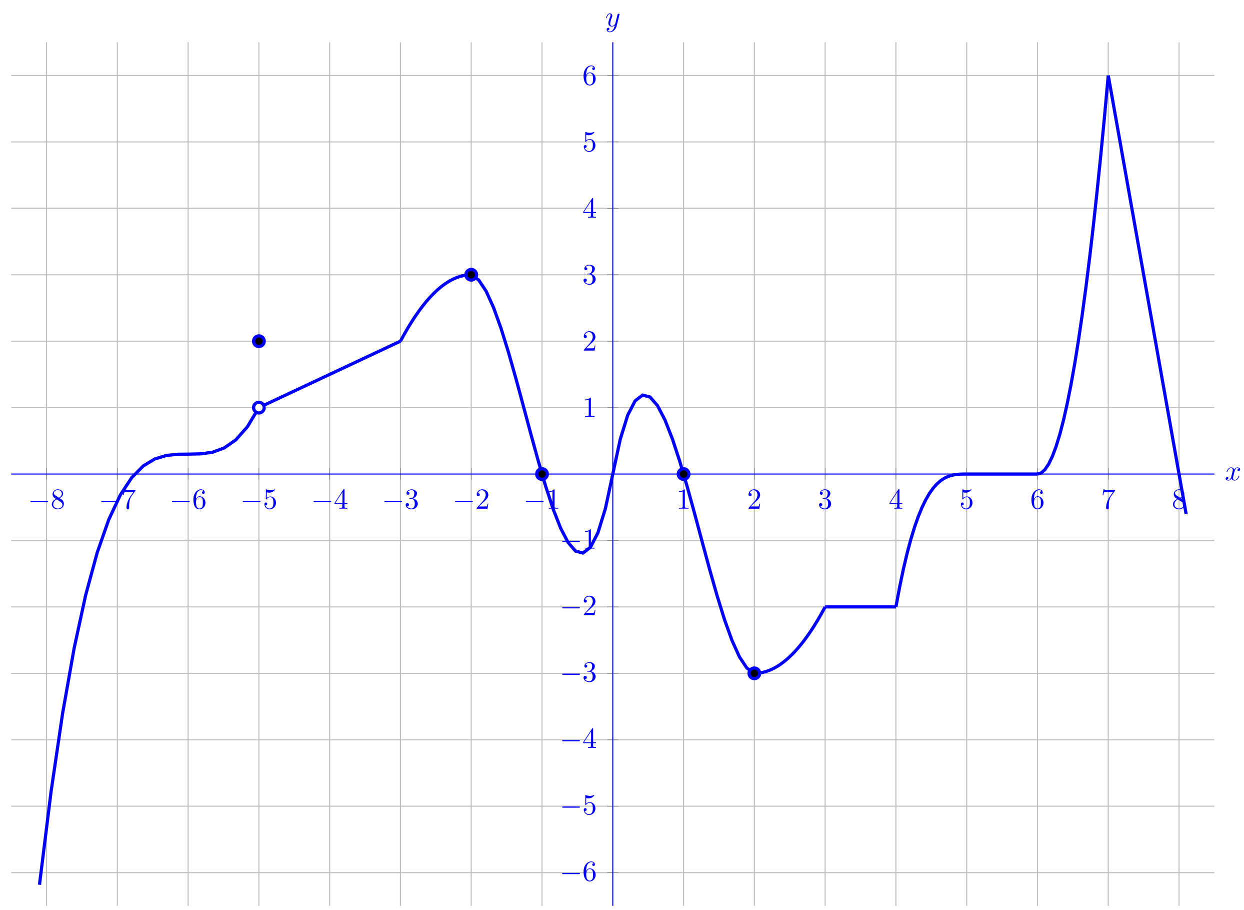
- The function $Q(x)$ is defined on $-8 \le x \le 8$.
- On the interval $(3, 8)$, the function $Q(x)$ is equal to the derivative of the function $h(x)$, which is shown in the graph at the right.



- $Q'(-6) = 0$ and $Q(x)$ is increasing in $-8 < x < -5$.
- $Q(x)$ is not continuous at $x = -5$ but $\lim_{x \to -5} Q(x)$ exists.
- $Q(-2) = 3$.
- $Q(x)$ has an x-intercept at $x = 1$.
- $Q(x) = -Q(-x)$ for $-3 < x < 3$.

Make sure that your graph is large and unambiguous.

Solution:



3. [6 points]

a. [4 points] For which value(s) of the constant A is the function

$$R(t) = \begin{cases} 5(13)^{At} & \text{for } t < 2. \\ 20 - 3t^2 & \text{for } t \geq 2. \end{cases}$$

continuous? Find your answer algebraically and give your answer in exact form. If no such value exists, write "DNE". Show all your work step by step.

Solution: The function $R(t)$ is continuous on $(-\infty, 2)$ and $(2, \infty)$. In order for $R(t)$ to be continuous at $t = 2$, $R(t)$ has to satisfy $R(2) = \lim_{t\to 2} R(t)$. Since $R(2) = 8 = \lim_{t\to 2^+} R(t)$, then it is only necessary that $\lim_{t\to 2^-} R(t) = \lim_{t\to 2^-} 5(13)^{At} = 5(13)^{2A} = 8$. This yields

$$5(13)^{2A} = 8$$
$$13^{2A} = \frac{8}{5} = 1.6$$
$$2A\ln(13) = \ln(1.6)$$
$$A = \frac{\ln(1.6)}{2\ln(13)}$$

Answer: $A = \dfrac{\ln(1.6)}{2\ln(13)}$

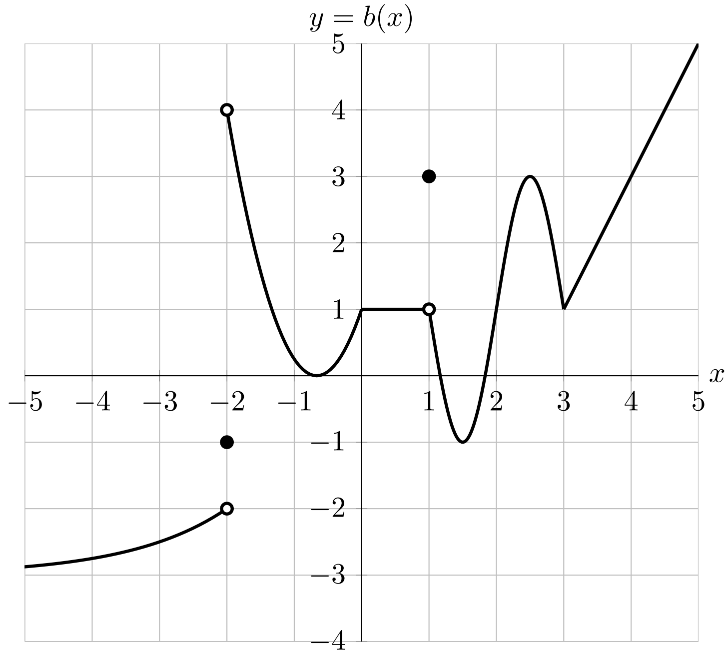
b. [2 points] A different function, $f(d)$, has the property that $\lim_{d\to\infty} f(d) = 10$. What is the value of $\lim_{d\to\infty} 4f(2d - 14) + 9$?

Write "DNE" if the limit does not exist or "NI" if there is not enough information to answer the question. You do not need to show your work.

Solution: Since $2d - 14$ tends to infinity as d tends to infinity, then $\lim_{d\to\infty} f(2d - 14) = 10$. Hence $\lim_{d\to\infty} 4f(2d - 14) + 9 = 4(10) + 9 = 49$.

Answer: 49

4. [14 points] A portion of the function $b(x)$ is depicted in the graph below. This function is defined for all real numbers x.



Find the exact value of the limits below. If any of the limits does not exists, write "DNE". If there is not enough information provided to you to answer the question, write "NI". You do not need to show your work.

a. [2 points] $\lim_{x\to -2} b(x)$

Solution: **Answer:** DNE

b. [2 points] $\lim_{x\to -2^-} b(x)$

Solution: **Answer:** -2

c. [2 points] $\lim_{t\to 1} b(t)$

Solution: **Answer:** 1

d. [2 points] $\lim_{m\to 0} \frac{b(4+m)-b(4)}{m}$

Solution: **Answer:** 2

e. [2 points] $\lim_{s\to 0^-} b(b(s))$

Solution: **Answer:** 1

f. [2 points] $\lim_{x\to 0^+} b(b(x))$

Solution: **Answer:** 3

g. [2 points] $\lim_{x\to\infty} b\left(-2+\frac{1}{x}\right)$

Solution: **Answer:** 4

5. [16 points] On a particularly cold winter day, Nia decides to turn on her gas powered heater at 5:00 pm. Over the next few hours, she records the temperature of her house and the amount of gas that the heater has used. She also notices that the temperature of her house seems to affect how loudly her dog barks. Nia uses the following three functions to model her observations:

- $T(t)$ represents the temperature (in degrees Fahrenheit) of Nia's house t minutes after 5:00 pm.
- $G(t)$ represents the amount of gas (in cubic feet) that the heater used during the first t minutes after 5:00 pm.
- $B(x)$ represents the sound intensity (in decibels) of the dog's barks when the temperature of her house is x degrees Fahrenheit.

You may assume that T, G and B are invertible and differentiable functions.

a. [10 points] Find a mathematical expression for each of the quantities below using the functions T,G,B, their inverses and/or their derivatives.

(i) The number of minutes the heater had been on when it had used 2 cubic feet of gas.

Solution:

Answer: $G^{-1}(2)$

(ii) The sound intensity (in decibels) of the dog's barks at 6:20 pm.

Solution:

Answer: $B(T(80))$

(iii) The approximate change in temperature of Nia's house (in degrees Fahrenheit) between 5:20 pm and 5:21pm.

Solution:

Answer: $T'(20)$

(iv) The amount of gas (in cubic feet) used by the heater between 5:30 pm and 7:00 pm.

Solution:

Answer: $G(120) - G(30)$

(v) The temperature (in degrees Fahrenheit) of Nia's house k minutes before 7:00 pm.

Solution:

Answer: $T(120 - k)$

b. [3 points] Complete the sentence below with a valid interpretation of the equation $T'(60) = 0.88$.

Solution:
In the first 15 seconds after 6:00 pm, the temperature of Nia's house **increases by about** $0.22°$ **F.**

This problem continues on the next page.

The statement of the problem has been included for your convenience.

On a particularly cold winter day, Nia decides to turn on her gas powered heater at 5:00 pm. Over the next few hours, she records the temperature of her house and the amount of gas that the heater has used.

Let $G(t)$ represent the amount of gas (in cubic feet) that the heater used during the first t minutes after 5:00 pm.

c. [3 points] Circle the <u>one</u> sentence that gives a valid interpretation of the equation

$$(G^{-1})'(3) = 72.$$

Solution:

(A) Nia's heater has used 3 cubic feet of gas at 6:12 pm.

(B) The amount of gas used by the heater between 5:03 pm and 5:04 pm will be approximately 72 cubic feet.

(C) Once the heater has used 72 cubic feet of gas, it takes about 3.6 minutes for it to use an additional 0.05 cubic feet of gas.

(D) It will take approximately 3.6 minutes for the amount of gas used by the heater to increase

(E) The heater uses 3 cubic feet of gas every 72 minutes.

6. [5 points] Let

$$A(w) = 5\sin(3w) - 4^{-w}.$$

Use the limit definition of the derivative to write an explicit expression for $A'(2)$. *Your answer should not involve the letter A. Do not attempt to evaluate or simplify the limit.* Please write your final answer in the answer box provided below.

Solution: A'(2) $= \lim_{h\to 0} \dfrac{5\sin(3(2+h)) - 4^{-(2+h)} - \left(5\sin(6) - 4^{-2}\right)}{h}$

7. [9 points] During the production of electricity from fossil fuel, nitrogen oxides are produced. The 1990 amendments to the Clean Air Act established the Acid Rain Program to reduce power plant nitrogen oxides and other emissions. A city in Michigan estimated that the annual nitrogen oxide emissions from its power plants were 191.4 thousand tons in 1990. Let $N(t)$ be a function that models the estimated annual nitrogen oxide emissions from that city's power plants (in thousand of tons) t years after 1970.
Find a formula for $N(t)$ assuming:

- The function $N(t)$ is continuous on its domain $[0, 47]$.
- The amount of annual nitrogen oxide emissions increased at a constant rate of 8 thousand tons every five years in between 1970 and 1990.
- The amount of annual nitrogen oxide emissions decayed exponentially by 20 percent every 3 years after 1990.

Solution:

- On $[0, 20]$: $N(t)$ is linear with rate of change $\frac{8}{5}$ thousands of tons of nitrogen oxide per year with $N(20) = 191.4$. Hence $N(t) = \frac{8}{5}(t - 20) + 191.4$.
- On $[20, 47]$: $N(t)$ is exponential. If $N(t) = ab^t$, then we know that $ab^{20} = 191.4$ and $ab^{23} = 191.4(0.8)$. Dividing both equations

$$\frac{ab^{23}}{ab^{20}} = b^3 \qquad \text{and} \qquad \frac{ab^{23}}{ab^{20}} = \frac{1.91.4(0.8)}{191.4} = 0.8.$$

Then $b^3 = 0.8$ yields $b = \sqrt[3]{0.8}$. Plugging into $ab^{20} = 191.4$ yields $a = 191.4(0.8)^{-\frac{20}{3}}$

$$N(t) = \begin{cases} \frac{8}{5}(t - 20) + 191.4 & \text{for} \quad 0 \le t < 20 \\ 191.4(0.8)^{\frac{1}{3}(t-20)} & \text{for} \quad 20 \le t \le 47 \end{cases}$$

8. [10 points] Let A and B be **positive** constants. The rational functions $y = P(x)$ and $y = Q(x)$ are given by the following formulas:

$$P(x) = \frac{5x(x-2)(Ax+1)^2}{(3x^2+B)(x^2-9)} \qquad Q(x) = \frac{P(x)(x-3)}{x-2}$$

Your answers below may depend on the constants A and B and should be in exact form. You do not need to show your work.

a. [3 points] Find the zeros of the function $y = P(x)$. If P has no zeros write "NONE".

Solution: Setting $5x(x-2)(Ax+1)^2 = 0$ you get $x = 0$, $x - 2 = 0$ and $(Ax+1)^2 = 0$. This yields $x = 0$, 2 and $-\frac{1}{A}$.

Answer: $x = 0$, 2 and $-\frac{1}{A}$.

b. [2 points] What is the domain of $P(x)$?

Solution: The only points not in the domain of $P(x)$ are the solutions to $(3x^2+B)(x^2-9) = 0$. Solving $x^2 - 9 = 0$ we get $x = \pm 3$. If we set $3x^2 + B = 0$, we get $x^2 = -\frac{B}{3} < 0$. This is not possible for any value of x. Then the only solutions are $x = \pm 3$.

Answer: $x \neq -3, 3$.

c. [2 points] Find the *equation(s)* of the horizontal asymptote(s) of $y = P(x)$. If it has no horizontal asymptotes, write "NONE".

Solution: To find the end behavior of $P(x)$ we need to find the leading coefficient of the numerator and the denominator. The leading term of the numerator is $5x(x-2)(Ax+1)^2$ is $5x(x)(Ax)^2 = 5A^2x^4$. The leading term of $(3x^2+B)(x^2-9)$ is $(3x^2)(x^2) = 3x^4$. Hence $\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{5A^2x^4}{3x^4} = \frac{5A^2}{3}$. This limit is the same as $\lim_{x\to-\infty} P(x)$.

Answer: $y = \frac{5A^2}{3}$

d. [3 points] If $A = 1$, find the values of c where $\lim_{x\to c} Q(x)$ does not exist. If no such values of c exist, write "NONE".

Solution: $\lim_{x\to c} Q(x)$ exists for all c in the domain of $Q(x) = \frac{5x(x-2)(Ax+1)^2(x-3)}{(3x^2+B)(x^2-9)(x-2)}$. Hence we need to check the limits at $c = -3$, 2 and 3. At $c = 2$, $\lim_{x\to 2} Q(x) = \frac{-2(2A+1)^2}{B+12}$ and at $c = 3$, $\lim_{x\to 3} Q(x) = \frac{15(3A+1)^2}{6(B+27)}$ (both of these points are holes in the graph of $Q(x)$). At $c = -3$, $Q(x)$ has a vertical asymptote hence $\lim_{x\to -3} Q(x)$ does not exist.

Answer: $c = -3$

9. [11 points] A group of marine biologists are studying life in the Challenger Deep, the deepest known point in the world's ocean. They use a special submarine to take samples of sea water for their study. Let $S(t)$ be the depth of the submarine (in miles) t minutes after it started collecting sea water samples. In this problem, depth will always be a positive number.

a. [5 points] Find a formula for $S(t)$ assuming that:

- $S(t)$ is a sinusoidal function.
- The submarine rises in 4 hours from a maximum depth of 6 miles to half a mile below the sea level (the closest point it gets to the surface).
- The submarine reaches its maximum depth 30 minutes after it starts taking sea water samples.

Solution: We know that $S(t)$ has a maximum at $t = 30$, then $S(t) = A\cos(B(t-30)) + k$ with $A > 0$. The amplitude of $S(t)$ is $\frac{6-0.5}{2} = 2.75$. The period of $S(t)$ is 480 (8 hours) and its midline is $y = \frac{6+0.5}{2} = 3.25$. Then $A = 2.75$, $B = \frac{2\pi}{480} = \frac{\pi}{240}$ and $k = 3.25$.

Answer: $S(t) = 2.75\cos\left(\frac{\pi}{240}(t-30)\right) + 3.25$

b. [6 points] During a second expedition, the depth of the submarine (in miles) is given by the function

$$D(t) = 3 + 2.5\cos\left(\frac{\pi}{90}t\right)$$

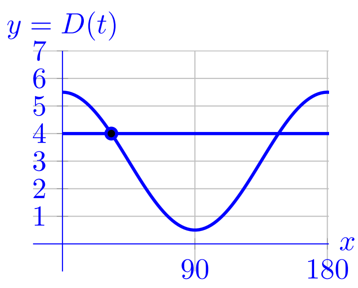
where t represents the time in minutes after the submarine started collecting samples. Once the submarine reaches a depth of 4 miles for the first time, how much time passes before it is at a depth of 4 miles for the second time? Your answer must be in exact form. Show all your work and include units.

Solution: Setting $D(t) = 4$ and solving for t:

$$3 + 2.5\cos\left(\frac{\pi}{90}t\right) = 4$$

$$\cos\left(\frac{\pi}{90}t\right) = 0.4$$

$$\frac{\pi}{90}t = \cos^{-1}(0.4) \qquad \text{then} \qquad t_1 = \frac{90}{\pi}\cos^{-1}(0.4).$$



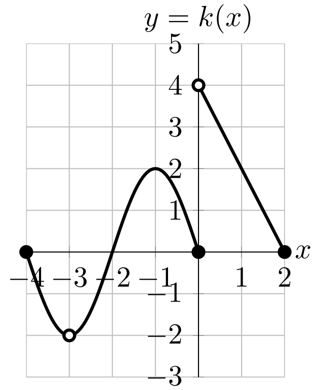
Answer: $180 - 2t_1 = 180 - \dfrac{180}{\pi}\cos^{-1}(0.4)$

A different approach using two consecutive solutions t_1 and t_2 where find t_2 from the equation

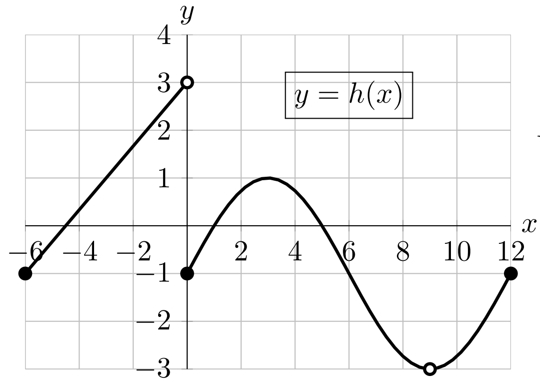
$$\frac{\pi}{90}t = 2\pi - \cos^{-1}(0.4) \qquad \text{then} \qquad t_2 = \frac{90}{\pi}(2\pi - \cos^{-1}(0.4)) = 180 - \frac{90}{\pi}\cos^{-1}(0.4)$$

yields **Answer:** $t_2 - t_1 = 180 - \dfrac{180}{\pi}\cos^{-1}(0.4)$

10. [6 points] The graph of the function $k(x)$ is shown below.



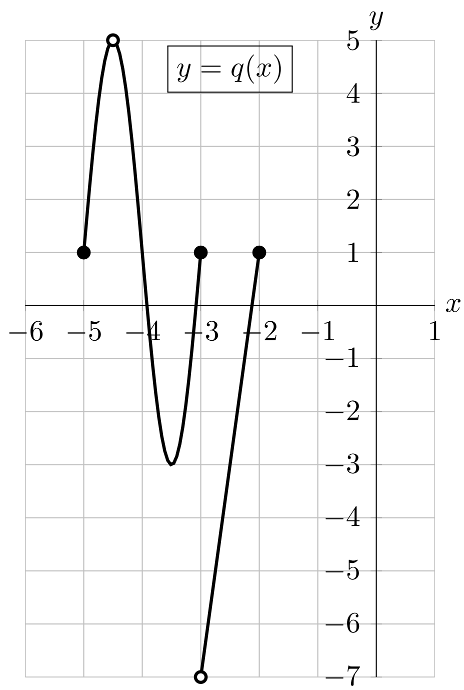
a. [3 points] The function $h(x)$ is obtained from $k(x)$ by one or more transformations and its graph is shown below. Note that the scale on the axes is not the same.



Write a formula for $h(x)$ in terms of the function k.

Solution: **Answer:** $h(x) = k\left(-\frac{1}{3}x\right) - 1$

b. [3 points] The function $q(x)$ is obtained from $k(x)$ by one or more transformations and its graph is shown below.



Which one of the following choices is the correct formula for $q(x)$?

Solution:

(A) $q(x) = 2k(-2(x+3)) - 3$

(B) $q(x) = 2k(-\frac{1}{2}x + 1) - 3$

(C) $q(x) = 2k(\frac{1}{2}(x+3)) - 6$

(D) $q(x) = -2k(2(x+4)) - 2$

(E) $\boxed{q(x) = -2k(2(x+3)) + 1}$

(F) $q(x) = -2k(2x+3) + 1$

(G) $q(x) = -2k(-\frac{1}{2}(x+4)) - 2$

(H) $q(x) = -2k(\frac{1}{2}(x-3)) + 1$

(I) NONE OF THESE