Math 115 — Second Midterm — November 12, 2018

EXAM SOLUTIONS

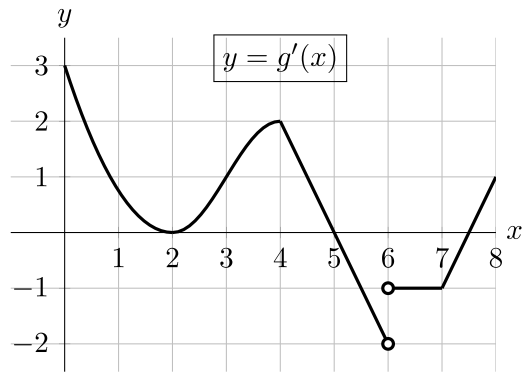
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 13 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	17	
2	6	
3	12	
4	12	
5	9	
6	4	

Problem	Points	Score
7	5	
8	7	
9	13	
10	7	
11	8	
Total	100	

1. [17 points] Let $g(x)$ and $h(x)$ be two functions. The graphs of $g'(x)$ and $h''(x)$ are shown below.

At right is the graph of $y = g'(x)$, the **derivative** of $g(x)$.
Assume that $g(x)$ is a **continuous** function.
Use the graph to answer the questions below. Circle all correct answers.



a. [2 points] At which of the following values of x is $g(x)$ not differentiable?

Solution:

$x = 2$ $\quad$ $x = 4$ $\quad$ $x = 5$ $\quad$ $\boxed{x = 6}$ $\quad$ $x = 7$ $\quad$ NONE OF THESE

b. [2 points] For which of the following values of x does $g(x)$ have a local maximum?

Solution:

$x = 2$ $\quad$ $x = 4$ $\quad$ $\boxed{x = 5}$ $\quad$ $x = 6$ $\quad$ $x = 7.5$ $\quad$ NONE OF THESE

c. [2 points] For which of the following values of x does $g(x)$ have an inflection point?

Solution:

$\boxed{x = 2}$ $\quad$ $x = 3$ $\quad$ $\boxed{x = 4}$ $\quad$ $x = 5$ $\quad$ $x = 7.5$ $\quad$ NONE OF THESE

d. [2 points] On which of the following intervals is $g(x)$ linear?

Solution:

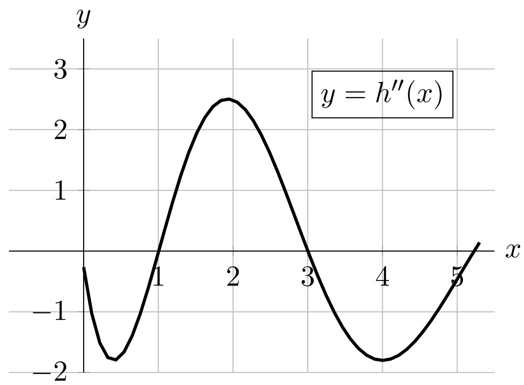
$(0, 2)$ $\quad$ $(4, 6)$ $\quad$ $\boxed{(6, 7)}$ $\quad$ $(6, 8)$ $\quad$ $(7, 8)$ $\quad$ NONE OF THESE

e. [2 points] For which of the following values of x does $g(x)$ attain a global maximum on the interval $[1, 7]$?

Solution:

$x = 2$ $\quad$ $x = 4$ $\quad$ $\boxed{x = 5}$ $\quad$ $x = 6$ $\quad$ $x = 7$ $\quad$ NONE OF THESE

Use the graph of $y = h''(x)$, the **second derivative** of $h(x)$, to answer the questions below. Circle all correct answers.



f. [2 points] Over which of the following intervals is $h(x)$ concave up on the entire interval?

Solution:

$(0, 1)$ $\boxed{(1, 3)}$ $(2, 4)$ $(4, 5)$ NONE OF THESE

g. [2 points] On which of the following intervals is the function $h'(x)$ (the derivative of $h(x)$) decreasing over the entire interval?

Solution:

$\boxed{(0, 1)}$ $(1, 3)$ $(2, 3)$ $\boxed{(4, 5)}$ NONE OF THESE

h. [3 points] If $h'(4) = 0$, which of the following statements must be true?

Solution:

A. $\boxed{x = 4 \text{ is a local maximum of } h(x).}$

B. $x = 4$ is a local minimum of $h(x)$.

C. $\boxed{x = 4 \text{ is an inflection point of } h'(x).}$

D. $\boxed{x = 4 \text{ is a critical point of } h(x).}$

E. $x = 4$ is an inflection point of $h(x)$.

F. NONE OF THESE

2. [6 points] Let A and B be constants and

$$k(x) = \begin{cases} 3x + \dfrac{B}{x} & \text{for} \quad 0 < x < 1 \\ \\ Bx^2 + Ax^3 & \text{for} \quad 1 \leq x. \end{cases}$$

Find the values of A and B that make the function $k(x)$ differentiable on $(0, \infty)$. Show all your work to justify your answers. If there are no such values of A and B, write NONE.

Solution: $k(x)$ will only be differentiable at $x = 1$ if it is also continuous at $x = 1$. In order for this to happen, we plug $x = 1$ in to both formulas of the original function and set them equal as well:

$$3 + B = B + A$$

From this second equation, we can subtract B from both sides to find $A = 3$.
The function $3x + \dfrac{B}{x}$ is differentiable on $(0, 1)$, and the function $Bx^2 + Ax^3$ is differentiable on $(1, \infty)$, so we just need values of A and B that will make $k(x)$ differentiable at $x = 1$.
We can compute the derivative:

$$k'(x) = \begin{cases} 3 - \dfrac{B}{x^2} & \text{for} \quad 0 < x < 1 \\ 2Bx + 3Ax^2 & \text{for} \quad 1 < x. \end{cases}$$

In order for $k(x)$ to be differentiable at $x = 1$, we must have

$$\begin{aligned} \lim_{h\to 0^-} \frac{k(1+h) - k(1)}{h} &= \lim_{h\to 0^+} \frac{k(1+h) - k(1)}{h} \\ \frac{d}{dx}(3x + \frac{B}{x})\Big|_{x=1} &= \frac{d}{dx}(Bx^2 + Ax^3)\Big|_{x=1} \\ 3 - \frac{B}{x^2}\Big|_{x=1} &= 2Bx + 3Ax^2\big|_{x=1} \\ 3 - B &= 2B + 3A \end{aligned}$$

In addition, Now we plug this value in for A in the earlier equation, giving us

$$3 - B = 2B + 9$$

Solving for B, we get $3B = -6$, so $B = -2$.

Answer: $A = 3$ $\qquad B = -2$

3. [12 points] Assume the function $h(t)$ is invertible and $h'(t)$ is differentiable. Some of the values of the function $y = h(t)$ and its derivatives are shown in the table below

t	0	1	2	3	4
$h(t)$	-2	2	3	4	8
$h'(t)$	3.5	0.5	2.5	1.5	5
$h''(t)$	6	0.25	0.3	-0.4	0.6

Use the values in the table to compute the exact value of the following mathematical expressions. If there is not enough information provided to find the value, write NI. If the value does not exist, write DNE. **Show all your work.**

a. [3 points] Let $a(t) = h(t^2 - 5)$. Find $a'(3)$.

Solution: Since $a'(t) = 2th'(t^2 - 5)$, then $a'(3) = 6h'(4) = 6(5) = 30$

Answer: 30

b. [3 points] Let $b(t) = \frac{h(t)}{t^2}$. Find $b'(4)$.

Solution: Since

$$b'(t) = \frac{h'(t)t^2 - 2th(t)}{t^4} \quad \text{then} \quad b'(4) = \frac{16h'(4) - 8h(4)}{256} = \frac{16(5) - 8(8)}{256} = \frac{16}{256} = \frac{1}{16}.$$

Answer: $\frac{1}{16}$

c. [3 points] Let $c(y) = h^{-1}(y)$. Find $c'(2)$.

Solution: Since $c'(y) = \frac{1}{h'(h^{-1}(y))}$ then $c'(2) = \frac{1}{h'(h^{-1}(2))} = \frac{1}{h'(1)} = 2.$

Answer: 2

d. [3 points] Let $g(t) = \ln(1 + 2h'(t))$. Find $g'(0)$.

Solution: Since $g'(t) = \frac{2h''(t)}{1 + 2h'(t)}$ then $g'(0) = \frac{2h''(0)}{1 + 2h'(0)} = \frac{2(6)}{1 + 2(3.5)} = \frac{12}{8} = 1.5$

Answer: 1.5

4. [12 points] Suppose $r(x)$ is a differentiable function defined for all real numbers x. The <u>derivative</u> and <u>second derivative</u> of $r(x)$ are given by

$$r'(x) = (x+1)^3(x-2)^{4/5} \qquad \text{and} \qquad r''(x) = \frac{(x+1)^2(19x-26)}{5(x-2)^{1/5}}$$

a. [6 points] Find the x-coordinates of all critical points of $r(x)$ and all values of x at which $r(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: The critical points occur at values of x where $r'(x) = 0$ or $r'(x)$ does not exist. $r'(x)$ exists for all values of x, so the only critical points are at points where

$$r'(x) = (x+1)^3(x-2)^{4/5} = 0.$$

That is when $x = -1$ and $x = 2$.
Now we need to know the sign of $r'(x)$ on three intervals. We will consider the sign of each factor of x in order to do this.

0	sign of $(x+1)^3$	sign of $(x-2)^{4/5}$	sign of $r'(x)$
$-\infty < x < -1$	$-$	$+$	$(-)(+) = -$
$-1 < x < 2$	$+$	$+$	$(+)(+) = +$
$2 < x < \infty$	$+$	$+$	$(+)(+) = +$

This tells us that $r(x)$ is decreasing for $x < -1$ and increasing for $x > -1$. Therefore we have a local minimum at $x = -1$ and no local maxes.

Answer: Critcal points at $x = -1, 2$

Local max(es) at $x =$NONE Local min(s) at $x = -1$

b. [6 points] Find the x-coordinates of all inflection points of $r(x)$. If there are none, write NONE. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: The only potential inflection points occur when $r''(x) = 0$ or $r''(x)$ does not exist. From the formula, we see that $r''(x) = 0$ when $x = -1$ and $x = \frac{26}{19}$, and $r''(x)$ does not exist at $x = 2$.
Again, we need to make a table of the sign of $r''(x)$ on intervals between these points, because inflection points only occur when $r(x)$ changes concavity, which happens when $r''(x)$ changes sign.

0	sign of $(x+1)^2$	sign of $(19x-26)$	sign of $(x-2)^{1/5}$	sign of $r''(x)$
$-\infty < x < -1$	$+$	$-$	$-$	$+$
$-1 < x < \frac{26}{19}$	$+$	$-$	$-$	$+$
$\frac{26}{19} < x < 2$	$+$	$+$	$-$	$-$
$2 < x < \infty$	$+$	$+$	$+$	$+$

Therefore, $r''(x)$ changes concavity at $x = \frac{26}{19}$ and $x = 2$.

Answer: Inflection points at $x = \frac{26}{19}, 2$

5. [9 points] In each of the following questions, draw a graph satisfying **all** the properties listed. There may be many correct answers. Make sure that your graph clearly shows all of the properties listed.

a. [5 points] The function $f(x)$ satisfies each of the following properties:

Solution:

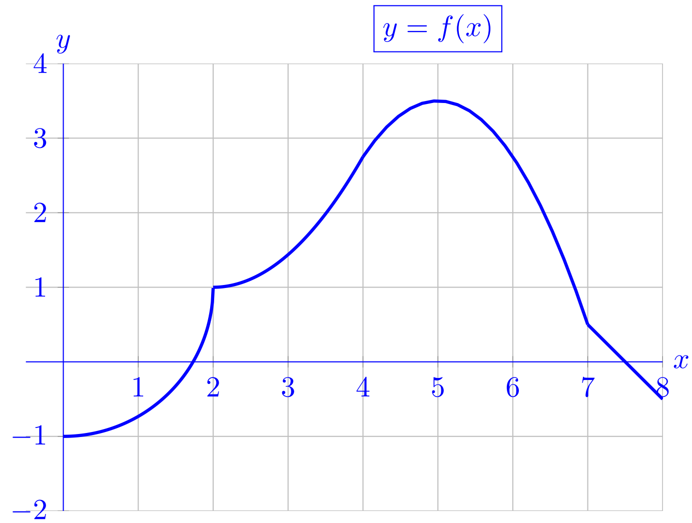
i) $f(x)$ is continuous on $(0, 8)$.

ii) $f(x)$ has a local maximum at $x = 5$.

iii) $f''(x) < 0$ on $(4, 7)$.

iv) $\lim_{x\to 2^-} f'(x) = \infty$ and

$\lim_{x\to 2^+} f'(x) = 0$



b. [4 points] The function $g(x)$ satisfies each of the following properties:

Solution:

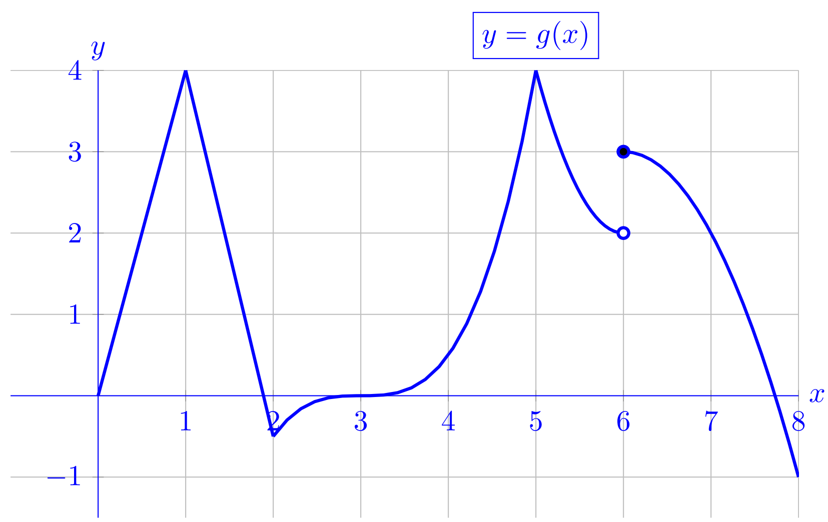
i) $g(x)$ is defined on $(0, 8)$.

ii) $g(x)$ has an inflection point at $x = 3$.

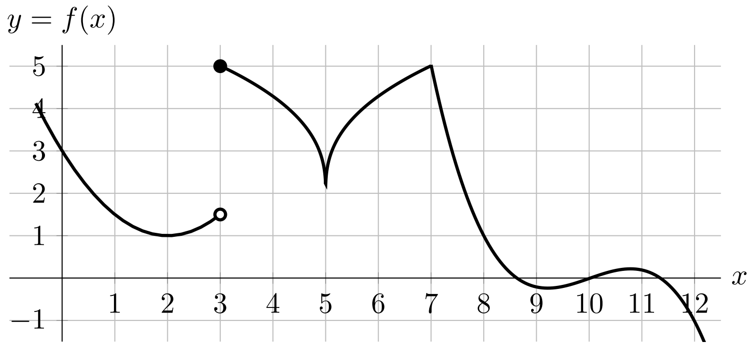
iii) $g(x)$ is discontinuous at $x = 6$.

iv) $g(x)$ has a local maximum at $x = 6$.

v) $g(x)$ has global maxima only at $x = 1$ and $x = 5$.



6. [4 points] The graph of the function $f(x)$ is shown below. Note that $f(x)$ has a vertical tangent line at $x = 5$.



a. [2 points] On which of the following intervals does the function $f(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle the correct answer(s).

[0,2]   [1,3]   [2,4]   [3,5]   NONE OF THESE

b. [2 points] On the interval $[8, 12]$ the hypotheses of the Mean Value Theorem are true for the function $f(x)$. What does the conclusion of this theorem say in this interval?

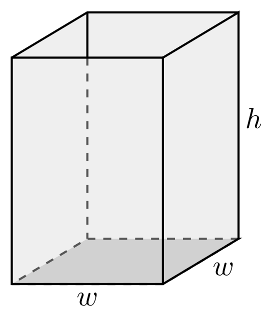
Answer:

Solution: There is some c on the interval $(8, 12)$ such that $f'(c) = \frac{f(12) - f(8)}{12 - 8} = -\frac{1}{2}$.

7. [5 points]

Yi is constructing a cardboard box. The base of the box will be a square of width w inches. The height of the box will be h inches. Yi will use gray cardboard for the sides of the box and brown cardboard for the bottom (the box does not have a top). Gray cardboard costs \$0.05 per square inch, while brown cardboard costs \$0.03 per square inch. Yi wants to spend \$20 on the cardboard for his box.

Write a formula for h in terms of w.



Solution: The area covered by the gray and the brown cardboard are $A_g = 4wh$ and $A_b = w^2$ respectively. Then the cost of the cardboard, in dollars, used in the cardboard is $C = 0.05A_g + 0,03A_b$. Hence w and h satisfy

$$C = 20 = 0.05(4wh) + 0.03w^2 = 0.2wh + 0.03w^2.$$

Then

$$h = \frac{20 - 0.03w^2}{0.2w}.$$

Answer: $h = \frac{20 - 0.03w^2}{0.2w}$

8. [7 points] Yi is working on a second box (not the one from the previous problem). This box will need a wire frame. The base of the box will be a square with width w inches. The height of the box will be h inches. Once the box is completed, its volume, V, in cubic inches will be

$$V = w^2 h.$$

Yi has to use thicker wire for the edges along the top and bottom of the box. Let M be the total mass in grams of the wire frame. The equation for M is

$$M = 18w + 6h.$$

Yi has a total of 540 grams of metal to make the wire frame. What values of w and h will maximize the volume of his box? Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the volume.

Solution: Solving for w in terms of h:

$$w = 30 - \frac{1}{3}h$$

$$V = \left(30 - \frac{1}{3}h\right)^2 h$$

$$V = \frac{1}{9}h^3 - 20h^2 + 900h$$

Now find the critical points of V.

$$\frac{dV}{dh} = \frac{1}{3}h^2 - 40h + 900$$

Critical points occur when $V' = 0$ so $h = 30$ or $h = 90$. Since w and h should both be positive, we get $h > 0$ and $30 - \frac{1}{3}h > 0$ so the possible values of h satisfy $0 < h < 90$. Since V is continuous on this interval and has only one critical point, $h = 30$, we only have to test the behavior of $V = \frac{1}{9}h^3 - 20h^2 + 900$ close to the endpoints $h = 0$ and $h = 90$ and the critical point $h = 30$.

$\lim_{h\to 0^+} V$	$h = 30$	$\lim_{h\to 90^-} V$
0	V(30)=12,000	0

Answer: The volume of the box is maximum when $w =$ 20 $h =$ 30

Other approach in this case (after finding the critical points of V in $0 < h < 90$): In $0 < h < 30$ and $30 < h < 90$ we pick $h = 20$ and $h = 50$ to find the sign of $\frac{dV}{dh}$ in these intervals. We get

$$\frac{dV}{dh}\Big|_{h=20} \approx 233.33 \quad \textit{and} \quad \frac{dV}{dh}\Big|_{h=50} \approx -266.66$$

	$0 < h < 30$	$30 < h < 90$
$\frac{dV}{dh}$	+	−

So, $h = 30$ is local max. Since V is continuous on this interval and has only one critical point, $h = 30$ is also the global max. When $h = 30$ we get $w = 20$.

9. [13 points] A curve $\mathcal{C}$ is defined implicitly by the equation

$$2x(y^2 - 4y) + 5x = 15.$$

Note that the curve $\mathcal{C}$ satisfies

$$\frac{dy}{dx} = \frac{-2y^2 + 8y - 5}{4xy - 8x}.$$

a. [4 points] Find all points on $\mathcal{C}$ with a vertical tangent line. Give your answers as ordered pairs (coordinates). Justify your answer algebraically. Write NONE if no such points exist.

Solution: $\mathcal{C}$ will have a vertical tangent line when the denominator of $\frac{dy}{dx}$ is 0, that is, $4xy - 8x = 0$. Factoring, we can rewrite this as $4x(y-2) = 0$. This happens when $x = 0$ or $y = 2$.
Now we need to find points on $\mathcal{C}$ where $x = 0$ or $y = 2$. Plugging $x = 0$ in to the equation for $\mathcal{C}$, we get $2 \cdot 0 \cdot (y^2 - 4y) + 5 \cdot 0 = 15$, which reduces to $0 = 15$. Since this is not true for any value of y, there are no points $(0, y)$ satisfying this equation.
Next we plug in $y = 2$. This gives

$$\begin{aligned} 2x \cdot (2^2 - 4 \cdot 2) + 5x &= 15 \\ 2x \cdot (-4) + 5x &= 15 \\ -3x &= 15 \\ x &= -5 \end{aligned}$$

So the denominator is 0 at the point $(x, y) = (-5, 2)$.
To make sure the numerator is not also 0, we plug $y = 2$ in to the numerator:

$$-2 \cdot 2^2 + 8 \cdot 2 - 5 = -8 + 16 - 5 = 3 \neq 0$$

Answer: $(-5, 2)$

b. [4 points] There is one point on $\mathcal{C}$ with a coordinate $(k, 0)$. Find the value of k and write the equation of the tangent line to $\mathcal{C}$ at the point $(k, 0)$. Your equation should not include the letter k.

Solution: To find the value of k, we plug $(k, 0)$ into the equation for $\mathcal{C}$.

$$\begin{aligned} 2 \cdot k(0^2 - 4 \cdot 0) + 5k &= 15 \\ 0 + 5k &= 15 \\ k &= 3 \end{aligned}$$

So $k = 3$. Now we plug $(3, 0)$ into the formula for $\frac{dy}{dx}$ to find the slope of the tangent line at this point: $\frac{-2 \cdot 0 + 8 \cdot 0 - 5}{4 \cdot 3 \cdot 0 - 8 \cdot 3} = \frac{5}{24}$. Finally, we use point-slope form to get the equation for the tangent line: $y = \frac{5}{24}(x - 3)$.

Answer: $k = 3$ Equation of tangent line: $y = \frac{5}{24}(x - 3)$

c. [5 points] Another curve $\mathcal{D}$ is defined implicitly by the equation

$$\cos(Wy^3 + Vxy) = \frac{1}{6}$$

where W and V are constants. Find a formula for $\frac{dy}{dx}$ in terms of x, y, W, and V. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution: Differentiating both sides with respect to x and then solving for $\frac{dy}{dx}$ gives

$$-\sin(Wy^3 + Vxy)(3Wy^2\frac{dy}{dx} + Vy + Vx\frac{dy}{dx}) = 0$$

$$(3Wy^2 + Vx)\frac{dy}{dx} + Vy = 0$$

$$\frac{dy}{dx} = \frac{-Vy}{3Wy^2 + Vx}$$

Alternatively, we can start by taking arccos of both sides, giving us $Wy^3 + Vxy = \arccos(\frac{1}{6})$, and then use implicit differentiation to get $(3Wy^2 + Vx)\frac{dy}{dx} + Vy = 0$; from here, we solve as in above.

Answer: $\frac{dy}{dx} = \frac{-Vy}{3Wy^2 + Vx}$

10. [7 points] The function g has the property that $g(x)$, $g'(x)$, and $g''(x)$ are defined for all real numbers. The quadratic approximation of $g(x)$ at $x = -2$ is

$$Q(x) = 4(x+2)^2 + \frac{1}{2}(x+2) - 5.$$

a. [5 points] Find the exact value of each of the following quantities. If there is not enough information to answer the question, write NI.

Solution:

$g(-2) =$ -5 $\qquad$ $g'(-2) =$ 0.5 $\qquad$ $g''(-2) =$ 8

$g(0) =$ NI $\qquad$ $Q'(0) =$ 16.5

b. [2 points] Write a formula for $L(x)$, the tangent approximation of $g(x)$ near $x = -2$.

Solution: The quadratic approximation near $x = -2$ satisfies $Q(x) = 4(x+2)^2 + L(x)$.

Answer: $L(x) = \frac{1}{2}(x+2) - 5$

11. [8 points] Consider the function

$$f(x) = \begin{cases} 4 - x - x^{\frac{2}{3}} & \text{for } -8 \le x \le 0 \\ 5xe^{-0.5x} + 4 & \text{for } x > 0. \end{cases} \quad \text{and its derivative} \quad f'(x) = \begin{cases} \dfrac{2 + 3x^{\frac{1}{3}}}{-3x^{\frac{1}{3}}} & \text{for } -8 < x < 0 \\ 5(1 - 0.5x)e^{-0.5x} & \text{for } x > 0. \end{cases}$$

Find the x-coordinates of the global maximum and the global minimum of the function $f(x)$ for $x \ge -8$. If one of them does not exist, write NONE in the answer line below. Use calculus to find your answers, and be sure to show enough evidence that the point(s) you find are indeed global extrema.

Solution: Critical points on the interval $(-8, 0)$ occur when the expression $\dfrac{2 + 3x^{\frac{1}{3}}}{-3x^{\frac{1}{3}}}$ is 0 or undefined. The numerator is equal to 0 at $x = -\frac{8}{27}$. The denominator is equal to zero when $x = 0$, which is not in the interval, so the only critical point in the interval $(-8, 0)$ is at $x = -\frac{8}{27}$.

Critical points on the interval $(0, \infty)$ occur when the expression $5(1 - 0.5x)e^{-0.5x}$ is 0 or undefined. Since $e^{-0.5x}$ is always positive, the only critical point is when $1 - 0.5x = 0$, which occurs when $x = 2$.

In addition to critical points, we must also check for global extrema by checking the end behavior for each piece of $f(x)$.

$$f(-8) = 4 - (-8) - (-8)^{2/3} = 12 - 4 = 8$$

$$f\left(-\frac{8}{27}\right) = 4 - \left(-\frac{8}{27}\right) - \left(-\frac{8}{27}\right)^{2/3} = 4 + \frac{8}{27} - \frac{4}{9} = \frac{104}{27}$$

$$f(0) = 4 - 0 - 0 = 4$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (5xe^{-0.5x} + 4) = 5 \cdot 0 \cdot 1 + 4 = 4$$

$$f(2) = 5 \cdot 2 \cdot e^{-1} + 4 = \frac{10}{e} + 4$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{5x}{e^{0.5x}} + 4 = 0 + 4 = 4$$

None of the end behavior is infinite. The highest number in the calculations above is 8, which occurs at $x = -8$. The lowest number is $\dfrac{104}{27}$, which occurs at $x = -\dfrac{8}{27}$

Answer:

Global maximum(s) at $x =$ -8

Global minimum(s) at $x =$ $-\frac{8}{27}$