Math 115 — Final Exam — Friday, December 14, 2018

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	10	
2	8	
3	14	
4	7	
5	7	
6	11	

Problem	Points	Score
7	11	
8	7	
9	9	
10	8	
11	8	
Total	100	

1. [10 points] Brianna is riding her unicycle on William Street. As she rides, she passes the Ann Arbor District Library. The function $u(t)$ represents Brianna's location (in meters west of the library) when she has been riding her unicycle for t seconds. The table below shows some values of $u'(t)$, the **derivative** of $u(t)$.

t	0	2	5	10	15	18	20	23	25	30
$u'(t)$	0	1	2	2.5	1.5	0	-1	-1.5	-2	-3

Note the following:

i) $u(23) = 2$.

ii) $u'(t)$ is continuous.

iii) $u'(t)$ satisfies:

- $u'(t)$ is increasing on $(0, 10)$.
- $u'(t)$ is decreasing on $(10, 30)$.

a. [2 points] Circle all of the following intervals on which $u(t)$ could be invertible.

Solution:

$\boxed{[3,8]}$ $\boxed{[2,15]}$ $[5, 20]$ $[10, 25]$ NONE OF THESE

b. [3 points] $u(t)$ is invertible on the interval $[20, 30]$. Let $f(t)$ be the inverse of $u(t)$ on that interval. Calculate $f'(2)$ and include units.

Solution:

$$f'(2) = \frac{1}{u'(u^{-1}(2))} = \frac{1}{u'(23)} = \frac{1}{-1.5} = -\frac{2}{3}.$$

Answer: $-\frac{2}{3}$ seconds per meter.

c. [2 points] Find the value of $\lim\limits_{x\to 23} \dfrac{u(x) - u(23)}{x - 23}$. If the limit does not exist, write DNE. If it cannot be determined based on the information given, write NI.

Solution:

Answer: $u'(23) = -1.5$.

d. [1 point] Estimate the value of $u''(24)$.

Solution:

$$u''(24) \approx \frac{-2 + 1.5}{2} = -0.25.$$

Answer: -0.25.

e. [2 points] Which of the following values of t could be inflection points of $u(t)$?

Solution:

5 $\boxed{10}$ 17 18 23 NONE OF THESE

2. [8 points] This problem is based on Brianna's ride with her unicycle. The statement of the previous problem is included for your convenience.

Brianna is riding her unicycle on William Street. As she rides, she passes the Ann Arbor District Library. The function $u(t)$ represents Brianna's location (in meters west of the library) when she has been riding her unicycle for t seconds.

The table below shows some values of $u'(t)$, the **derivative** of $u(t)$.

t	0	2	5	10	15	18	20	23	25	30
$u'(t)$	0	1	2	2.5	1.5	0	-1	-1.5	-2	-3

Note the following:

i) $u(23) = 2$.

ii) $u'(t)$ is continuous.

iii) $u'(t)$ satisfies:

- $u'(t)$ is increasing on $(0, 10)$.
- $u'(t)$ is decreasing on $(10, 30)$.

a. [4 points] Use a right Riemann sum with 4 subintervals of equal size to estimate Brianna's displacement between times $t = 10$ and $t = 30$. Write all the terms in your sum. Include units.

Solution:

Right Riemann sum $= 5(u'(15)+u'(20)+u'(25)+u'(30)) = 5(1.5-1-2-3) = -22.5$ meters.

b. [1 point] Is your answer in part **a** an overestimate or an underestimate? Circle your answer. If there is not enough information circle NI.

Solution:

OVERESTIMATE [✓ UNDERESTIMATE] NEITHER NI

c. [3 points] Which of the following *must* be equal to Brianna's average velocity during the time interval $[15, 20]$? Circle all correct answers.

Solution:

A. $\boxed{\dfrac{u(20) - u(15)}{20 - 15}}$

B. $\dfrac{u'(15) + u'(20)}{2}$

C. $\dfrac{u'(15) + u'(18) + u'(20)}{3}$

D. $\dfrac{1}{20 - 15} \displaystyle\int_{15}^{20} u(t)dt$

E. $\boxed{\dfrac{1}{20 - 15} \displaystyle\int_{15}^{20} u'(t)dt}$

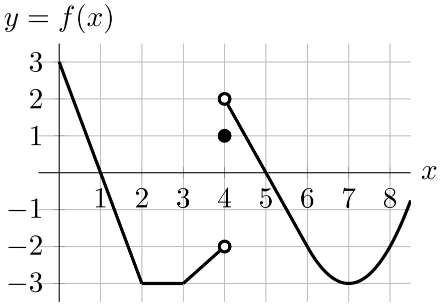
F. $\dfrac{1}{18 - 15} \displaystyle\int_{15}^{18} u'(t)dt + \dfrac{1}{20 - 18} \displaystyle\int_{18}^{20} u'(t)dt$

G. $\boxed{\dfrac{1}{20 - 15} \left(\displaystyle\int_{0}^{20} u'(t)dt - \displaystyle\int_{0}^{15} u'(t)dt \right)}$

H. $\dfrac{1}{20 - 0} \displaystyle\int_{0}^{20} u'(t)dt - \dfrac{1}{15 - 0} \displaystyle\int_{0}^{15} u'(t)dt$

I. NONE OF THESE

3. [14 points] Suppose $f(x)$ is an even function. A piece of the graph of $f(x)$ is given below. Note that $f(x)$ is piecewise linear for $0 \leq x \leq 6$. Find the following quantities. If any of their values do not exist, write DNE. If there is not enough information to answer, write NI.



a. [1 point] Find $\lim_{p \to 4^+} f(p)$.

Solution:

Answer: 2

b. [2 points] Find $\lim_{m \to 0} \dfrac{f(1+m) - f(1)}{m}$.

Solution:

Answer: -3

c. [3 points] Let $g(x) = \dfrac{1}{\sqrt{4 + f(2x)}}$. Find $g'(2.5)$.

Solution:

$$g'(x) = -\frac{1}{2}(4 + f(2x))^{-\frac{3}{2}}(2f'(2x)) = -\frac{f'(2x)}{(4 + f(2x))^{\frac{3}{2}}} \qquad g'(2.5) = -\frac{f'(5)}{(4 + f(5))^{\frac{3}{2}}} = -\frac{(-2)}{4^{\frac{3}{2}}} = \frac{1}{4}$$

Answer: $\dfrac{1}{4}$

d. [3 points] Recall that $f(x)$ is even. Find $\int_{-3}^{1} (5f(t) - 3)\, dt$.

Solution:

$$\int_{-3}^{1} (5f(t) - 3)\, dt = 5\left(\int_{-3}^{-1} f(t))\, dt + \int_{-1}^{1} f(t))\, dt\right) - \int_{-3}^{1} 3dt$$

$$= 5(-4.5 + 3) - 12 = -7.5 - 12 = -19.5.$$

Answer: -19.5

e. [3 points] Let $j(x)$ be an antiderivative of $f(x)$ with $j(5) = 3$. Suppose that $p(x)$ is the quadratic approximation of $j(x)$ near $x = 5$. Find a formula for $p(x)$.

Solution: We know that $p(x) = j(5) + j'(5)(x-5) + \dfrac{j''(5)}{2}(x-5)^2$. Since $j(x)$ be an antiderivative of $f(x)$ then $j'(5) = f(5) = 0$ and $j''(5) = f'(5) = -2$. Hence

Answer: $p(x) = 3 - (x-5)^2$.

f. [2 points] Find all the values of a with $-3 \leq a \leq 3$ such that $\int_{-2}^{a} f(x)\, dx = 0$.

Solution:

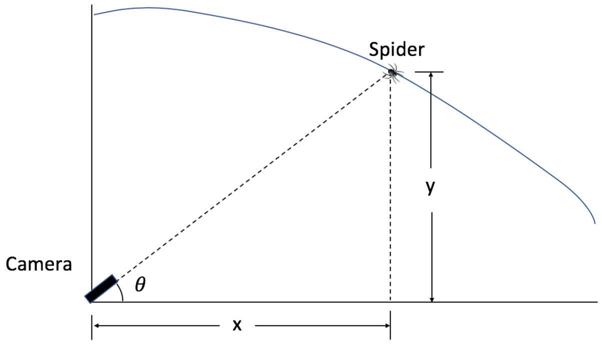
Answer: $a = -2, 0, 2$

4. [7 points] Casey is making a documentary about the wildlife that lives in a local cave. She found a spider of a new species climbing down along the ceiling of the cave (as shown in the diagram below). Here

- x is the spider's distance to the right, in ft, of the camera
- y is the height, in ft, of the spider from the ground
- θ is the angle, in radians, made by the ground and the line joining Casey's camera and the spider.

The camera is following the spider as it walks along the ceiling of the cave. **Find the rate at which the angle θ is changing** when the following conditions hold:

- The spider is 10 ft above the ground.
- The spider's distance to the right of the camera is increasing at 0.4 feet per second.
- The spider's height is decreasing at a rate of 0.2 feet per second.
- The angle $\theta = \frac{\pi}{6}$ radians.



Use the equation

$$\tan(\theta) = \frac{y}{x}.$$

satisfied by the variables x, y and θ to find your answer. Include units. Show all your work.

Solution: Taking derivatives with respect to time we get

$$\frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}.$$

Solving for $\frac{d\theta}{dt}$ we get

$$\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}\cos^2(\theta)$$

We are given $y = 10$, $\frac{dx}{dt} = 0.4$, $\frac{dy}{dt} = -0.2$ and $\theta = \frac{\pi}{6}$ then $\tan(\theta) = \frac{y}{x}$ yields $\frac{10}{x} = \frac{1}{\sqrt{3}}$ or $x = 10\sqrt{3}$. Using these values into this equation, we get

$$\frac{d\theta}{dt} = \frac{10\sqrt{3}(-0.2) - 10(0.4)}{(10\sqrt{3})^2}\cos^2\left(\frac{\pi}{6}\right) = \frac{-2\sqrt{3} - 4}{300}\left(\frac{3}{4}\right) = \frac{-\sqrt{3} - 2}{200}$$

Answer: $\frac{-\sqrt{3} - 2}{200}$ radians per second.

5. [7 points] Consider the family of functions $f(x) = bx^5e^{cx}$ with parameters b and c. Note that

$$f'(x) = bx^4e^{cx}(cx+5) \quad \text{and} \quad f''(x) = bx^3e^{cx}(c^2x^2+10cx+20)$$

a. [2 points] Find all values of b and c that make

$$\lim_{x\to\infty} f(x) = \infty \quad \text{AND} \quad \lim_{x\to-\infty} f(x) = 0.$$

Solution: Conditions for b: $b > 0$ Conditions for c: $c > 0$

b. [5 points] Suppose $b > 0$ and $c > 0$. Find the critical point(s) of $f(x)$ and the x-coordinates of the local extrema of $f(x)$. Your answer must be in exact form and may be expressed in terms of the constants b and c. You should use calculus to find and justify your answers. For each answer blank below, write NONE if appropriate.

Solution: There are no points where $f'(x)$ is undefined, so all of the critical points can be found by solving $f'(x) = bx^4e^{cx}(cx+5) = 0$. This yields $x = 0$ and $x = -\frac{5}{c}$. We test the critical point $x = -\frac{5}{c}$ using the Second Derivative Test

$$f''\left(-\frac{5}{c}\right) = b\left(-\frac{5}{c}\right)^3 e^{-5}(c^2\left(-\frac{5}{c}\right)^2 + 10c\left(-\frac{5}{c}\right) + 20) = 625e^{-5}\frac{b}{c^3} > 0.$$

Hence $x = -\frac{5}{c}$ is a local minimum.

We use the First Derivative Test to classify $x = 0$. We need to test on the intervals $\left(-\frac{5}{c}, 0\right)$ and $(0, \infty)$. We use the points $x = -\frac{4}{c}$ and $x = 1$.

$$f'\left(-\frac{4}{c}\right) = b\left(-\frac{4}{c}\right)^4 e^{-4}\left(c\left(-\frac{4}{c}\right)+5\right) = b\left(\frac{256}{c^4}\right)e^{-4} > 0.$$

$$f'(1) = be^c(c+5) \qquad \text{then} \qquad f'(1) = +(+)(+) = +.$$

Hence $x = 0$ is neither a local maximum or a local minimum.

Critical point(s) at $x = 0$ and $x = -\frac{5}{c}$

Local max(es) NONE Local min(s) $x = -\frac{5}{c}$

6. [11 points] Ben has recently acquired a cabbage press and is opening a business selling cabbage juice. Let $R(x)$ and $C(x)$ be the revenue and cost, in dollars, of selling and producing x cups of cabbage juice. Ben only has resources to produce up to a hundred cups. After some research, Ben determines that

$$R(x) = 6x - \frac{1}{40}x^2 \qquad \text{for} \qquad 0 \leq x \leq 100$$

and

$$C(x) = \begin{cases} 60 + 2x & 0 \leq x \leq 20 \\ 70 + 1.5x & 20 < x \leq 100. \end{cases}$$

a. [3 points] What is the smallest quantity of juice Ben will need to sell in order for his profit to not be negative? Round your answer to the nearest hundredth of a cup. Show your work.

Solution: We consider values of x such that $R(x) = C(x)$. We first look in $[0, 20]$

$$60 + 2x = 6x - \frac{1}{40}x^2 \qquad \text{or} \qquad \frac{1}{40}x^2 - 4x + 60 = 0.$$

Using the quadratic formula we get $x = 80 \pm 20\sqrt{10}$. Only one of these two solutions, $x = 80 - 20\sqrt{10} \approx 16.75$, is in the interval $[0, 20]$.

The last step is to verify that $R(x) - C(x)$ is negative on the interval $[0, \quad 80 - 20\sqrt{10})$ and positive on the interval $(80 - 20\sqrt{10}, \quad 20]$. We can test this by picking points in each interval. For example, $R(0) - C(0) = -60$ and $R(20) - C(20) = 10$. **Answer:** 16.75 cups.

For the following parts, determine how many cups of cabbage juice Ben needs to sell in order to maximize the given quantity. If there is no such value, write NONE. Use calculus to find and justify your answers.

b. [3 points] Ben's revenue.

Solution: The critical points of $R(x)$ can be found by solving $R'(x) = 6 - \frac{1}{20}x = 0$. This occurs when $x = 120$ which is not in $[0, 100]$. Hence the maximum has to be at one of the endpoints $x = 0$ or $x = 100$. Since $R(0) = 0$ and $R(100) = 350$, the maximum revenue is attained at $x = 100$. **Answer:** 100 cups.

c. [5 points] Ben's profit.

Solution: Since $P(20) = 10$ and

$$\lim_{x \to 20^-} P(x) = \lim_{x \to 20^-} 6x - \frac{1}{40}x^2 - (60 + 2x) = 10$$

and

$$\lim_{x \to 20^+} P(x) = \lim_{x \to 20^+} 6x - \frac{1}{40}x^2 - (70 + 1.5x) = 10.$$

Then $P(x)$ is continuous on $[0, 20]$. The critical points of $P(x)$ can be found by solving $P'(x) = R'(x) - C'(x) = 0$ in the intervals $(0, 20)$ and $(20, 100)$.

- On $(0, 20)$ we need to solve $6 - \frac{1}{20}x = 2$. This yields $x = 80$ (outside the interval).
- On $(20, 100)$ we need to solve $6 - \frac{1}{20}x = 1.5$. This yields $x = 90$.

Hence the critical points of $P(x)$ are $x = 0$ and $x = 90$. Since $P(x)$ is continuous then the global maximum must lie on the critical points or in the endpoints.

x	0	20	90	100
$P(x)$	-60	10	132.5	130

Answer: 90 cups.

7. [11 points] Zavier the zoo-keeper is breeding fish in a large aquarium. As the population of fish increases, he notices that the amount of waste in the bottom of aquarium is also increasing. Each week he measures how much waste has accumulated at the bottom of the aquarium. The function $W(t)$ models the amount of waste, in millimeters, at the bottom of the aquarium t weeks after Zavier began his measurements. Note the following information about the function $W(t)$.

- $W(t)$ is continuous on the interval $[0, 9]$.
- During the first 3 weeks, the amount of waste increases exponentially from 102.4 mm at time $t = 0$ to 200 mm at time $t = 3$.
- After 3 weeks, Zavier buys several catfish to eat the waste at the bottom of the aquarium. Over the next 6 weeks, the **rate of change** in the amount of waste (in millimeters per week) is given by the function $g(t) = t^2 - 12t + 26$.

a. [8 points] Write a piecewise defined formula for the **continuous** function $W(t)$ on the interval $[0, 9]$. Show all your work.

Solution:

- On $0 \leq t \leq 3$: We have $W(t) = ab^t$ with $W(0) = 102.4$ and $W(3) = 200$. Hence $a = 102.4$ and $W(3) = 102.4b^3 = 200$. Solving for b we get $b = \left(\frac{200}{102.4}\right)^{\frac{1}{3}}$. Then $W(t) = 102.4\left(\frac{200}{102.4}\right)^{\frac{1}{3}t}$ in $0 \leq t \leq 3$.
- On $3 < t \leq 9$: Since $W'(t) = g(t)$, then $W(t) = \frac{1}{3}t^3 - 6t^2 + 26t + C$ where C is a constant that we have to determine. We know that $W(t)$ is continuous at $t = 3$, then

$$\lim_{t\to 3^+} W(t) = \lim_{t\to 3^+} \frac{1}{3}t^3 - 6t^2 + 26t + C = 33 + C = W(3) = 200.$$

This yields that $C = 200 - 33 = 167$. Hence

$$W(t) = \begin{cases} 102.4\left(\dfrac{200}{102.4}\right)^{\frac{1}{3}t} & \text{if} \quad 0 \leq t \leq 3 \\ \dfrac{1}{3}t^3 - 6t^2 + 26t + 167 & \text{if} \quad 3 < t \leq 9. \end{cases}$$

b. [3 points] Zavier continues recording data and notices that, after 50 weeks, the amount of waste is modeled by the function

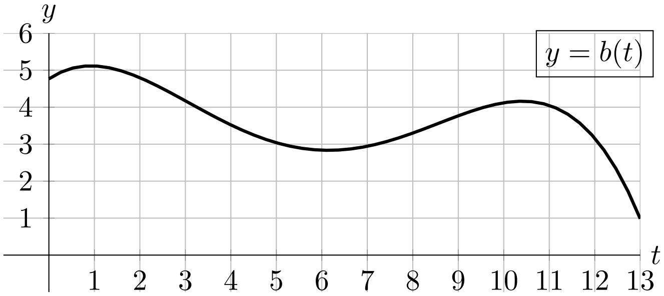
$$Q(t) = \frac{4(a-t)^2(bt-2)^2}{(3t^2+7t+10)(4-t)(-2t+c)}$$

where a, b, and c are positive constants. What happens to the amount of waste in the long run? Circle the correct answer and fill in the blank if necessary. Your answer may include the constants a, b or c.

Solution:

i) It increases without limit.
ii) It approaches zero.
iii) $\boxed{\text{It approaches a positive limit with value } L \text{ where } L = \frac{2b^2}{3}}$
iv) None of these.

8. [7 points] Ben buys cabbage for his juice business. Let $b(t)$ be the rate at which Ben buys cabbage, in pounds per month, for his business t months after the beginning of 2015. The graph of $b(t)$ is shown below.



a. [3 points] Ben already has bought 100 lbs of cabbage at the beginning of 2015. Write a mathematical expression involving the function b, its derivative and/or a definite integral that represents the total number of pounds of cabbage Ben bought by the end of 2015.

Solution:

Answer: $100 + \int_0^{12} b(t)dt$

b. [2 points] Let $A(t)$ be the amount of cabbage, in pounds, Ben has bought during the first t months of 2015. Suppose $A(5) = 120$. Find a formula for the tangent line approximation $L(t)$ of $A(t)$ near $t = 5$.

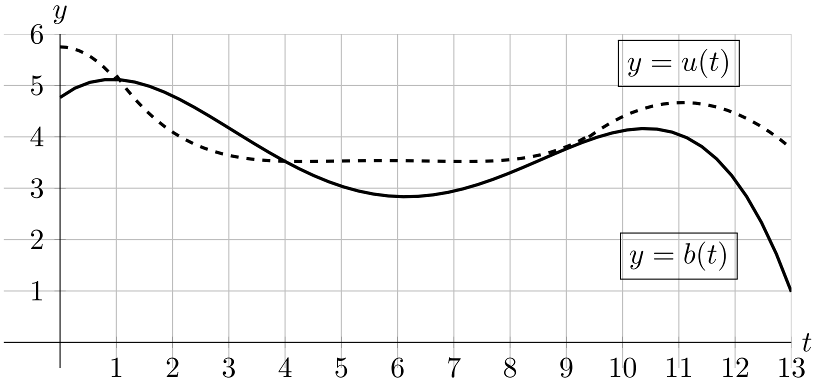
Solution: $L(t) = 120 + 3(t - 5)$

c. [2 points] Which of the following must be true? Circle your answer.

Solution:

$L(4.5) < A(4.5)$ [✓ $L(4.5) > A(4.5)$] $L(4.5) = A(4.5)$ NOT ENOUGH INFORMATION

9. [9 points] Recall from the last problem that $b(t)$ is the rate at which Ben buys cabbage, in pounds per month, for his business t months after the beginning of 2015. Let $u(t)$ be the rate at which Ben uses the cabbage he buys, in pounds per month, t months after the beginning of 2015. The graphs of the functions $b(t)$ (solid line) and $u(t)$ (dashed line) are shown below.



Let $h(t)$ be the amount of cabbage, in pounds, that Ben bought but has not used for his business. In questions **a**, **b** and **c**, answer NONE when appropriate. You do not need to justify your answers.

a. [2 points] Find and classify all local extrema of $h(t)$ in $0 < t < 13$.

Solution: Local max(es) at $t = 4$ Local min(s) at $t = 1$

b. [2 points] Find all global extrema of $h(t)$ in $0 \leq t \leq 13$.

Solution: Global max(es) at $t = 4$ Global min(s) at $t = 13$

c. [2 points] Estimate all inflection points of $h(t)$ in $0 < t < 13$.

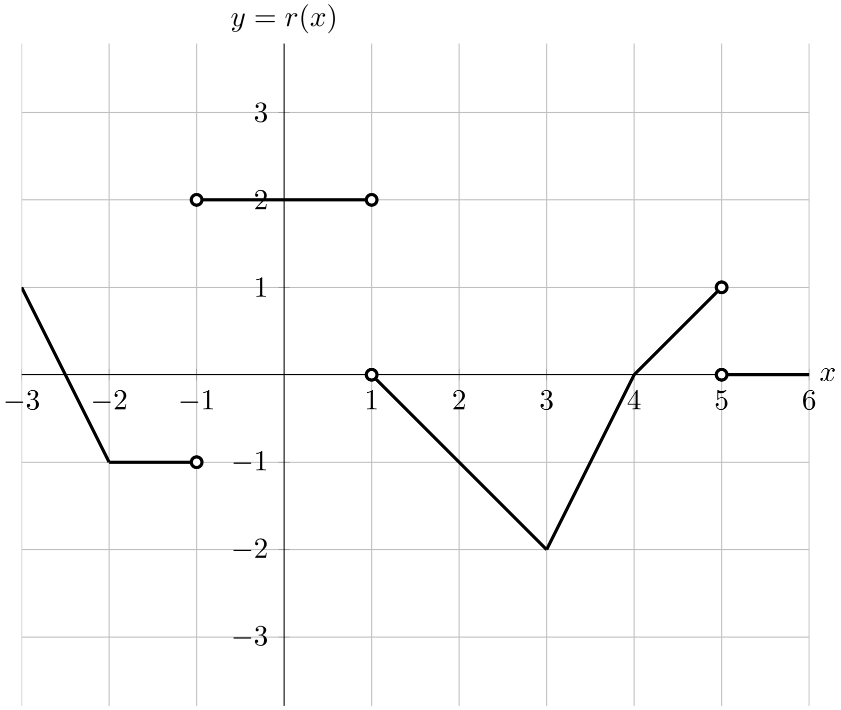
Solution:

Inflection point(s) at $t \approx 2.2$, 6.2 and 9.2.

d. [3 points] Complete the following sentence to give a practical interpretation of $h'(14.5) = -1.3$.

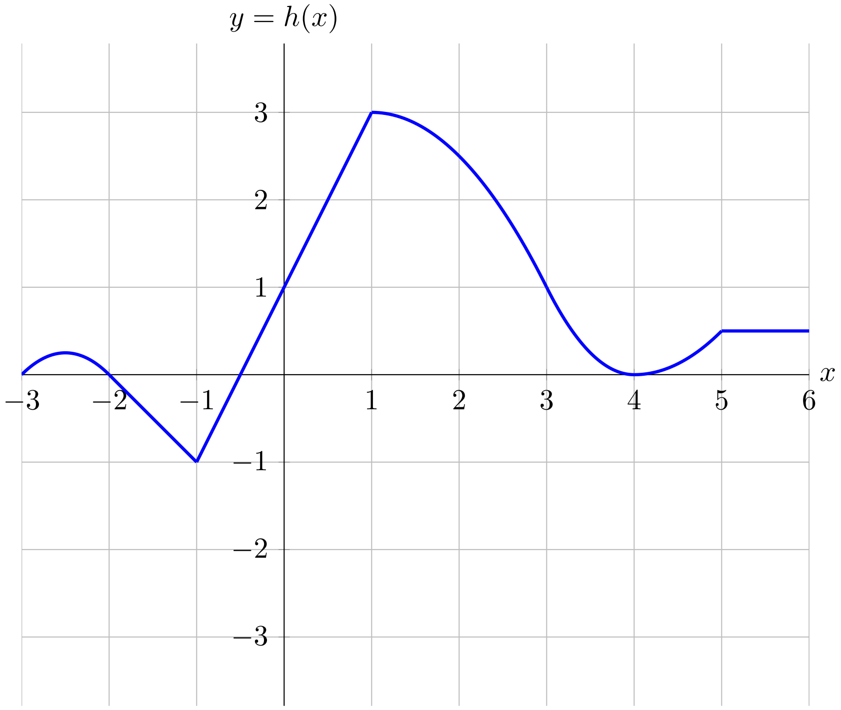
Solution:
During the first half of March of 2016, the amount of cabbage that Ben has bought but not used for his business **decreases by approximately 0.65 pounds.**

10. [8 points] Part of the graph a piecewise-linear function $r(x)$ is shown below.

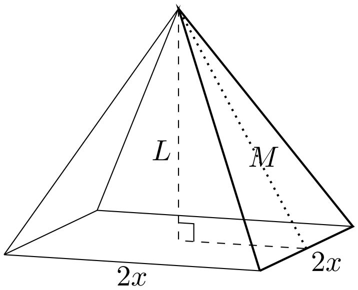


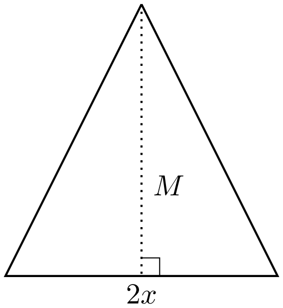
The function $h(x)$ is a continuous antiderivative of $r(x)$ with $h(0) = 1$. On the axes below, sketch the graph of $h(x)$ over the interval $-3 \leq x \leq 6$. Make sure to pay attention to:

- where h is and is not differentiable.
- where h is increasing/decreasing/constant.
- where h is linear/concave up/concave down.
- the values of $h(-3), h(-2), h(-1), ..., h(5), h(6)$.



11. [8 points] Jose is building a pyramid-shaped hat with 4 triangular sides of the same shape. Each side has a base of $2x$ centimeters. The height of the hat is L centimeters. Each of the four triangular sides has height M centimeters (see the diagram below).





a. [3 points] Jose plans to use 400 square centimeters of material in the construction of the hat. Find a formula for the height L of the hat only in terms of x. *Your formula should not include the letter M.* Show all your work.

Solution: Using Pythagorean theorem $L = \sqrt{M^2 - x^2}$. Since he uses 400 square centimeters of material in the construction of the hat, we have $4Mx = 400$. This yields

$$L = \sqrt{M^2 - x^2} = \sqrt{\left(\frac{100}{x}\right)^2 - x^2}$$

Answer: $L(x) = \sqrt{\left(\frac{100}{x}\right)^2 - x^2}$.

b. [2 points] The volume of a pyramid is given by $V = \frac{1}{3}Ah$, where A is the area of the base and h is the height of the pyramid. Find a formula for the volume of the hat V, in cubic centimeters, in terms only of the variable x. *Your answer should not include the variables L and/or M.*

Solution: **Answer:** $V(x) = \frac{4}{3}x^2\sqrt{\left(\frac{100}{x}\right)^2 - x^2}$

c. [3 points] What is the domain of the function $V(x)$ in the context of this problem?

Solution: We need to find where $y(x) = \left(\frac{100}{x}\right)^2 - x^2 > 0$ when $x > 0$ (since x is a length). First we find where

$$\begin{aligned}\left(\frac{100}{x}\right)^2 - x^2 &= 0\\ \frac{10,000}{x^2} &= x^2\\ x^4 &= 10,000 \qquad \text{this implies} \qquad x = \pm 10.\end{aligned}$$

We need to test the sign of $y(x)$ (and hence $V(x)$) on $(0, 10)$ and $(10, \infty)$. The fact that $y(2) = 50^2 - 4 > 0$ and $y(50) = 4 - 50^2 < 0$ shows the domain of $V(x)$ is $0 < x < 10$.

Answer: $0 < x < 10$