Math 115 — First Midterm — October 7, 2019

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is not permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form.* Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	12	
3	10	
4	12	
5	12	

Problem	Points	Score
6	4	
7	9	
8	12	
9	10	
10	11	
Total	100	

1. [8 points] The Amazing Wanda is performing a magic act.

a. Let $V(t)$ be the volume, in decibels (dB), of the audience's applause t seconds after the beginning of the act.

i. [2 points] At time $t = 0$, the audience is already clapping at a volume of 52 dB. During Wanda's first trick, which lasts 45 seconds, the volume of the audience's applause increases at a constant rate of 0.4 dB per second. Write a formula for the function $V(t)$ during the first trick.

Answer: $V(t) =$ $0.4t + 52$ for $0 \le t \le 45$

ii. [4 points] During Wanda's second trick, which begins at $t = 45$ and lasts until the end of the act at time $t = 95$, the volume of the audience's applause increases by 1.2% every second. Write a piecewise formula for the function $V(t)$ on the interval $[0, 95]$. Make sure that $V(t)$ is a continuous function.

Solution: The first piece of the formula for $V(t)$ was the answer to part i.

The function $V(t)$ is exponential for $45 \le t \le 95$ with growth factor 1.012. Note that at time $t = 45$, Wanda's audience is clapping at a volume of $0.4(45) + 52 = 70$ dB.

Approach 1: Since the constant percent growth begins at time $t = 45$, we need to shift the exponential function with initial value 70 forward (to the right) by 45 seconds. This gives

$$70(1.012)^{t-45} = 70(1.012)^{-45}(1.012)^t = \frac{70}{1.012^{45}}(1.012)^t.$$

Note that this answer (which is exact) can be approximated by either of the following

$$70(1.012)^{t-45} \approx 40.924(1.012)^t \text{ or } 70(1.012)^{t-45} \approx 40.924e^{0.0119t}$$

Approach 2: We can also find a formula for the exponential piece of $V(t)$ by using the facts that $V(45) = 70$ and $V(t)$ is continuous. Solving for a in the resulting equation $70 = a(1.012)^{45}$. gives the value $a = \frac{70}{1.012^{45}}$ resulting in the same formula as the previous approach.

Answer: $$V(t) = \begin{cases} 0.4t + 52 & \text{for } 0 \le t \le 45 \\ 70(1.012)^{t-45} & \text{for } 45 < t \le 95 \end{cases}$$

b. [2 points] A few minutes after her act, Wanda returns to the stage for an encore performance. Let $W(s)$ be the volume, in dB, of the audience's applause s seconds after the encore begins. A table of some values of $W(s)$ is given below.

s	0	2	3
$W(s)$	3.00	3.60	4.32

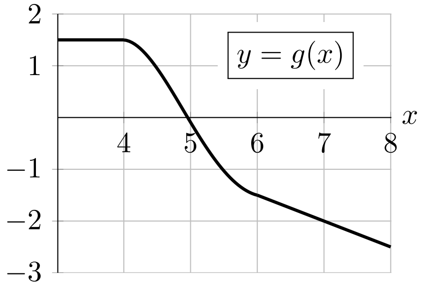
Could $W(s)$ be an exponential function? Circle your answer below. Show your work to justify your answer.

Solution: No, $W(s)$ could not be exponential. Over the interval $[2, 3]$, the function grows by a factor of $4.32/3.60 = 1.2$. If $W(s)$ were exponential, it would therefore have to grow by a factor of $(1.2)^2$ over the interval $[0, 2]$. But $3.6/3 = 1.2 \neq (1.2)^2$.

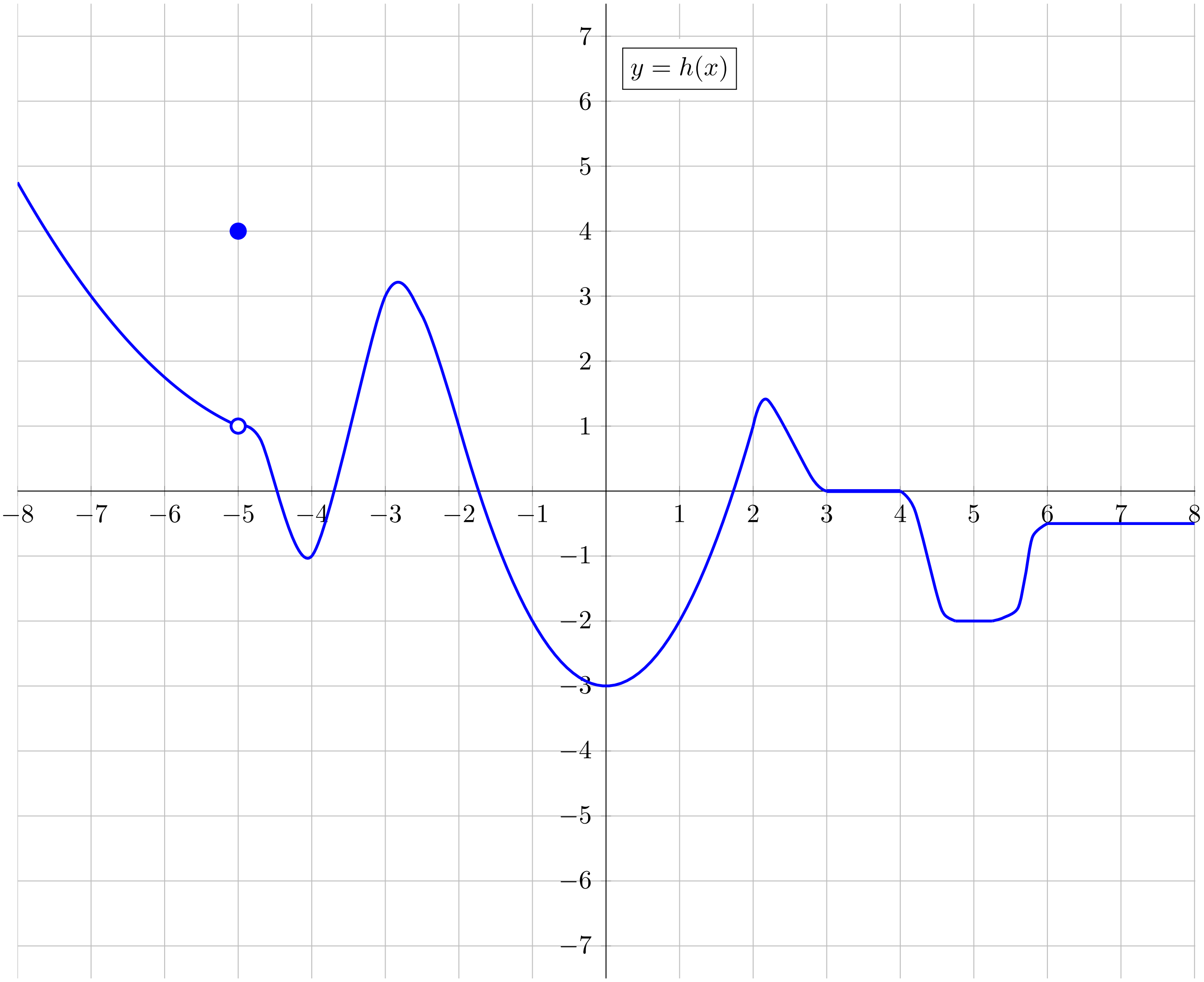
Answer: YES [✓ NO]

2. [12 points] On the axes provided below, sketch the graph of a single function $h(x)$ that satisfies all of the following conditions.

- The domain of the function $h(x)$ includes $-8 < x < 8$.
- $h(x)$ is concave up and decreasing on $-8 < x < -5$.
- $\lim_{x\to -5^-} h(x) = 1$ and $\lim_{x\to -5^+} h(x) = 1$.
- $h(x)$ is <u>not</u> continuous at -5.
- $\dfrac{h(-4) - h(-2)}{-4 - (-2)} = 1$.
- $h(x)$ has a y-intercept of -3.
- $h(x) = h(-x)$ for $-2 \le x \le 2$.
- On the interval $3 < x < 8$, the function $h(x)$ is the derivative of the function $g(x)$, which is shown in the graph to the right.



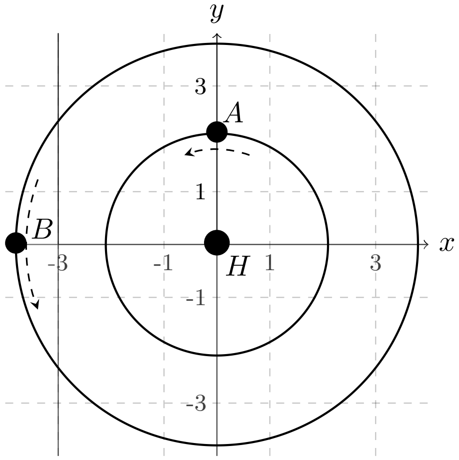
Solution: One possible solution is shown below.



3. [10 points] Horatio the Daring is performing a dangerous stunt. Helicopter A and Helicopter B are circling around Horatio to film the event. Let t be the amount of time, in minutes, since the start of Horatio's stunt.

A top-down view of the flight paths is shown at right. The locations of the helicopters at $t = 0$ are labeled A and B, respectively, and Horatio's location is labeled H (at the origin).

All distances are measured in kilometers (km). The helicopters are flying counter-clockwise around Horatio in perfect circles at a constant height above the ground.



a. [4 points] Helicopter A moves at a constant speed of 0.7 km/min around a circle of radius 2.1 km. Write a formula for the function $a(t)$ that gives the y-coordinate of Helicopter A at time t.

Solution: A circle of radius 2.1 km has circumference 4.2π km. If the helicopter flies at a constant speed of 0.7 km/min, then it takes the helicopter $(4.2\pi)/0.7 = 6\pi$ minutes to fly all the way around the circle. The period of $a(t)$ is therefore 6π minutes.

Note that the y-coordinate of Helicopter A begins at its maximum value, so we can model $a(t)$ using a cosine function without a horizontal shift.

Answer: $a(t) =$ $2.1\cos\left(\frac{t}{3}\right)$ **or** $2.1\sin\left(\frac{t}{3}+\frac{\pi}{2}\right) = 2.1\sin\left(\frac{1}{3}\left(t+\frac{3\pi}{2}\right)\right)$

b. [6 points] The x-coordinate of Helicopter B at time t is given by the formula

$$b(t) = -3.8\cos\left(\frac{\pi}{32}t\right).$$

Find **<u>all</u>** values of t during the first hour of the stunt at which the location of Helicopter B has x-coordinate less than or equal to -3. Give your answer as one or more intervals, with endpoints in <u>exact form</u>.

Solution: First we find the times in the first hour ($0 \le t \le 60$) when $b(t) = -3$. We know

$$-3.8\cos\left(\frac{\pi}{32}t\right) = -3$$

$$\cos\left(\frac{\pi}{32}t\right) = \frac{3}{3.8},$$

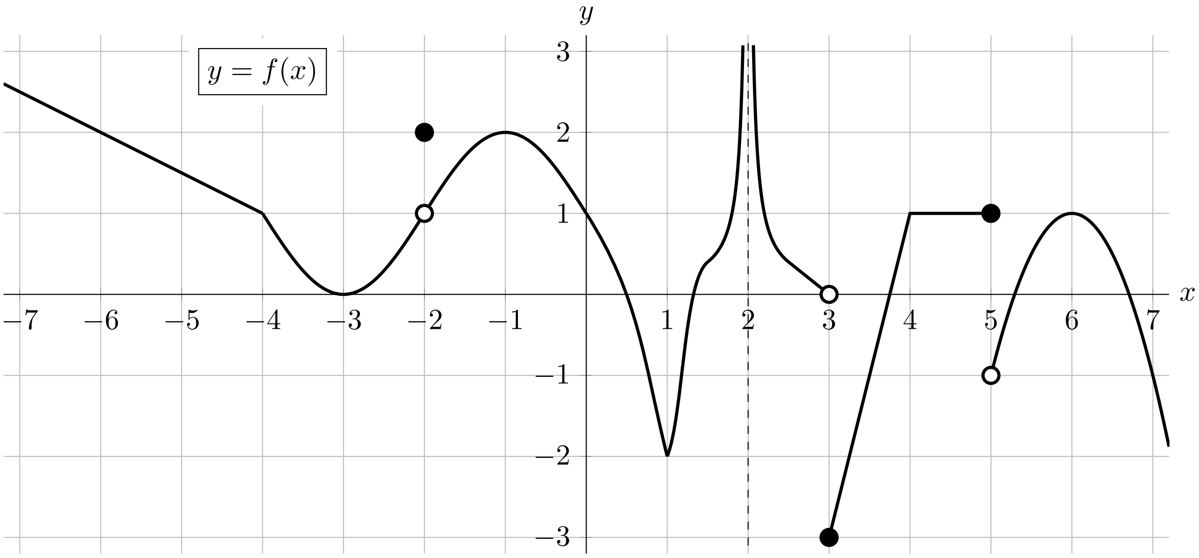
so one solution is given by $\frac{\pi}{32}t = \arccos\left(\frac{3}{3.8}\right)$

which gives $t = \frac{32}{\pi}\arccos\left(\frac{3}{3.8}\right) \approx 6.73$ min.

The period of $b(t)$ is $2\pi/(\pi/32) = 64$ min, and because $b(t)$ begins at its minimum value, we can use symmetry to see that the second (and final) time at which $b(t) = -3$ during the first hour is $t = 64 - \frac{32}{\pi}\arccos\left(\frac{3}{3.8}\right) \approx 57.27$ minutes.
The x-coordinate of Helicopter B is greater than -3 between these two times.

Answer: $\left[0, \frac{32}{\pi}\arccos\left(\frac{3}{3.8}\right)\right]$ **and** $\left[64 - \frac{32}{\pi}\arccos\left(\frac{3}{3.8}\right), 60\right]$

4. [12 points] A portion of the graph of a function f is shown below. Note that $f(x)$ has a vertical asymptote at $x = 2$.



Throughout this problem, you do not need to show work or explain your reasoning.

For parts **a.** and **b.** below, circle **<u>all</u>** of the listed values satisfying the given statement. If there are no such values listed, circle NONE.

a. [2 points] For which of the following values of a is $f(x)$ continuous at $x = a$?

[✓ $a = -3$] $a = -2$ [✓ $a = 1$] $a = 3$ NONE

b. [2 points] For which of the following values of b is $\lim_{x \to b^+} f(x) = f(b)$?

[✓ $b = -4$] $b = -2$ [✓ $b = 0$] [✓ $b = 3$] NONE

In the following parts, evaluate each of the given quantities. If the value does not represent a real number (including the case of limits that diverge to ∞ or $-\infty$), write "DNE" or "does not exist."

c. [2 points] $\lim_{x \to -2} f(x)$

Answer: 1

d. [2 points] $\lim_{x \to 5} f(x)$

Answer: **DNE**

e. [2 points] $\lim_{x \to 2} e^{-f(x)}$

Answer: 0

f. [2 points] $\lim_{h \to 0} \frac{f(-6+h) - f(-6)}{h}$

Answer: $-\frac{1}{2}$

5. [12 points] A weather balloon is launched and heads straight up away from the ground. Let $R(t)$ be the height, in kilometers, of the balloon above the ground t minutes after its launch. The function $R(t)$ is invertible and differentiable.

t	1	3	9	18	35	45	60	63	86
$R(t)$	0.01	0.19	0.4	0.84	2.3	3	3.7	4.1	8.9

a. [2 points] On which of the following intervals could $R(t)$ be concave up on the entire interval? Circle **all** correct answers.

$[1,9]$ [✓ $[3,18]$] [✓ $[9,35]$] NONE OF THESE

b. [2 points] Find the balloon's average velocity between times $t = 3$ and $t = 18$. Show work and *include units.*

Solution: The balloon's average velocity over this time period is given by

$$\frac{R(18)-R(3)}{18-3} = \frac{0.84-0.19}{18-3} = \frac{0.65}{15} = \frac{13}{300}.$$

Answer: $\frac{0.84-0.19}{18-3} = \frac{13}{300} \approx 0.0433$ **km/min**

c. [3 points] Estimate the balloon's instantaneous velocity at $t = 63$. Show work and *include units.*

Solution: The balloon's instantaneous velocity at $t = 63$ is $R'(63)$.
The closest given time to $t = 63$ is $t = 60$, so we use the average rate of change of R over $[60, 63]$ to estimate $R'(63)$. (Note that $t = 86$ is very far from $t = 63$ when compared to $t = 60$.)

$$R'(63) \approx \frac{R(63)-R(60)}{63-60} = \frac{4.1-3.7}{63-60} = \frac{0.4}{3} = \frac{4}{30} \approx 0.133 \text{ km/min}$$

Answer: **approximately** 0.133 **km/min**

d. [3 points] Estimate $(R^{-1})'(3)$. Show work and *include units.*

Solution: The closest given distances to 3 km are 2.3 km and 3.7 km. We will estimate $(R^{-1})'(3)$ by taking the average rate of change of R^{-1} over the interval $[2.3, 3.7]$.

$$(R^{-1})'(3) \approx \frac{R^{-1}(3.7)-R^{-1}(2.3)}{3.7-2.3} = \frac{60-35}{3.7-2.3} = \frac{25}{1.4} = \frac{125}{7} \approx 17.86 \text{ min/km}$$

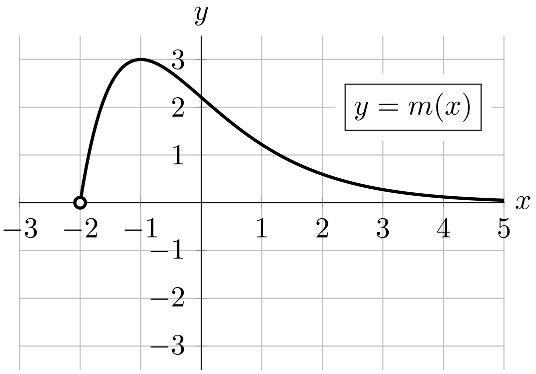
It would also be okay to use the interval $[2.3, 3]$ or $[3, 3.7]$, which would give about 14.3 and 21.4 min/km, respectively. The average of these two estimates would give the estimate we found above.

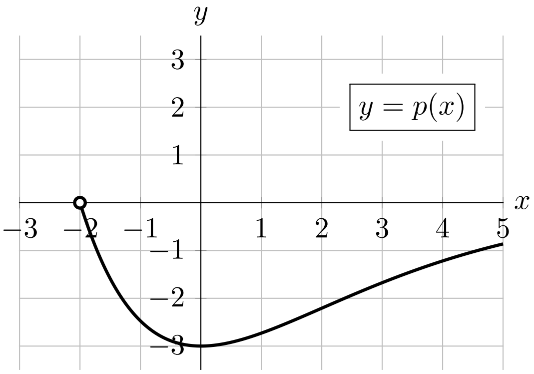
Answer: $(R^{-1})'(3) \approx$ 17.86 **min/km**

e. [2 points] Let $M(s)$ be the height, in meters, of the balloon above the ground s seconds after its launch. Find a formula for $M(s)$ in terms of R and s. (There are 1000 meters in one kilometer.)

Answer: $M(s) = 1000R\left(\frac{1}{60}s\right)$

6. [4 points] Shown below at left is a portion of the graph of a function $m(x)$. Shown below at right is a portion of the graph of a function $p(x)$, which can be obtained from $m(x)$ through one or more graph transformations. Find a formula for $p(x)$ in terms of $m(x)$.





Answer: $p(x) = -m\left(\frac{1}{2}x - 1\right) = -m\left(\frac{1}{2}(x-2)\right)$

7. [9 points] For a constant c, let

$$K(x) = \frac{2^{cx}}{e^{x-c}}.$$

a. [5 points] Use the limit definition of the derivative to write an explicit expression for $K'(3)$. *Your answer may include the constant c but should not involve the letter K. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $K'(3) = \lim_{h \to 0} \frac{\frac{2^{c(3+h)}}{e^{(3+h)-c}} - \frac{2^{c(3)}}{e^{3-c}}}{h}$

b. [4 points] Find the value of c so that $K(1) = 5$. Give your answer in **exact form** and show all your work.

Solution: We want c such that

$$\frac{2^{c(1)}}{e^{1-c}} = 5, \text{ or}$$
$$2^c = 5e^{1-c}.$$

Solving, we find that

$$\begin{aligned} \ln\left(2^c\right) &= \ln\left(5e^{1-c}\right) \\ \ln\left(2^c\right) &= \ln(5) + \ln\left(e^{1-c}\right) \\ c\ln(2) &= \ln(5) + 1 - c \\ c\ln(2) + c &= \ln(5) + 1 \\ c(\ln(2) + 1) &= \ln(5) + 1 \\ c &= \frac{\ln(5)+1}{\ln(2)+1}. \end{aligned}$$

Answer: $c = \frac{\ln(5)+1}{\ln(2)+1}$

8. [12 points] The size of the harvest at a kale farm is a function of the total amount of compost the farm uses in the fields.

- Let $K(c)$ be the size (as measured by weight) of the farm's kale harvest, in tons, when the farm uses c cubic meters (m^3) of compost.
- Let $P(h)$ be the farm's profit, in thousands of dollars, when their kale harvest is h tons.

The functions $K(c)$ and $P(h)$ are differentiable, and the function $P(h)$ is invertible.

a. [2 points] Using a complete sentence, give a practical interpretation of the equation

$$P^{-1}(86) = 53.$$

Solution: In order for the kale farm to make 86 *thousand dollars in profit, they need to harvest* 53 *tons of kale.*

b. [3 points] Write a single equation involving K, P, and/or P^{-1} that represents the following statement.

If the farm uses 1125 m^3 *of compost, their profit will be twice as large as if they had used* 700 m^3 *of compost.*

Answer: $P(K(1125)) = 2P(K(700))$

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$K'(950) = 0.2.$$

If the farm uses 955 m^3 *of compost instead of* 950 m^3, ...

Solution: ...they would harvest roughly 1 *additional ton of kale.*

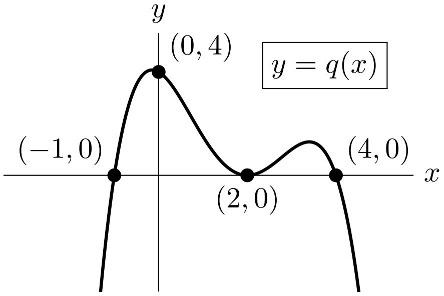
d. [4 points] Write a single equation involving the derivative function(s) K', P', and/or $(P^{-1})'$ that represents the following statement.

In order for the farm's profit to be \$101,500 *rather than* \$100,000*, their kale harvest must be about* 0.9 *tons larger.*

Answer: $(P^{-1})'(100) = 0.6$

9. [10 points] Parts **a.** – **c.** below are not related. You do not need to show work on this page, but partial credit may be earned for work shown.

a. [4 points] A portion of the graph of a polynomial function $q(x)$ is shown below. Find a possible formula for $q(x)$ of the smallest possible degree. Assume that all of the key features of the graph are shown.



Solution: We see that the degree of $q(x)$ must be even. The zeros of $q(x)$ are -1, 2, and 4. Note that 2 is a double zero. We use the point $(0, 4)$ to find the leading coefficient.

$$\begin{aligned} q(x) &= C(x+1)(x-2)^2(x-4) \\ 4 &= C(0+1)(0-2)^2(0-4) \\ -\frac{1}{4} &= C \end{aligned}$$

Answer: $q(x) = -\frac{1}{4}(x+1)(x-2)^2(x-4)$

b. [3 points] Find the formula for a rational function $r(x)$ that has a hole with an x-value of 5, a vertical asymptote at $x = 1$, and a horizontal asymptote at $y = -2$.

Solution: Note that the factor of x in the numerator could be replaced by any degree 1 factor $(x + B)$ with B a constant.

Answer: $r(x) = \frac{-2(x-5)x}{(x-5)(x-1)}$

c. [3 points] Consider the function

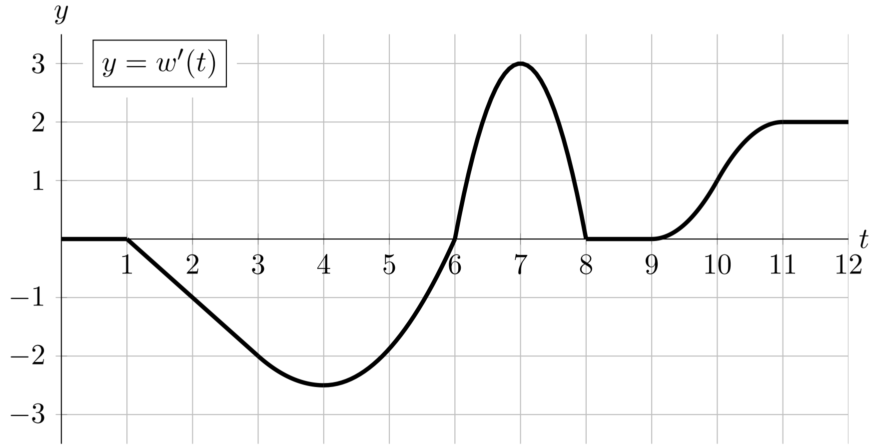
$$z(x) = \frac{4^{-x} - 2x^2}{15x + 3x^2}.$$

Find $\lim_{x\to\infty} z(x)$ and $\lim_{x\to-\infty} z(x)$. If the value does not represent a real number (including the case of limits that diverge to ∞ or $-\infty$), write "DNE" or "does not exist."

Solution: Note that $4^{-x} = \left(\frac{1}{4}\right)^x$, so $\lim_{x\to\infty} 4^{-x} = 0$ and $\lim_{x\to-\infty} 4^{-x}$ does not exist (as 4^{-x} diverges to ∞ as $x \to -\infty$).

Answer: $\lim_{x\to\infty} z(x) = -\frac{2}{3}$ and $\lim_{x\to-\infty} z(x) =$ **DNE**

10. [11 points] Let $w(t)$ be the amount of water, in cubic meters (m^3), in a small pond t hours after noon on a certain summer day. The function $w'(t)$, the **derivative** of $w(t)$, is graphed below.



a. [3 points] At 10 PM, is the amount of water increasing or decreasing? Circle your answer below. At what rate? *Include units.*

Answer: [✓ INCREASING] DECREASING at a rate of: $1\ \mathrm{m}^3/\mathrm{hr}$

b. [2 points] Over which of the following intervals of t, if any, is the amount of water in the pond constant? Circle **all** correct answers.

[✓ $[0, 1]$] $[1, 3]$ $[11, 12]$ NONE OF THESE

c. [2 points] Over which of the following intervals of t, if any, is the amount of water in the pond decreasing at a constant rate? Circle **all** correct answers.

$[0, 1]$ $[1, 3]$ $[11, 12]$ [✓ NONE OF THESE]

d. [2 points] At which of the following times t is the amount of water in the pond increasing the fastest? Circle the **one** correct answer.

$t = 4$ $t = 6.3$ [✓ $t = 7$] $t = 10$

e. [2 points] At which of the following times t does the pond contain the least amount of water? Circle the **one** correct answer.

$t = 0$ $t = 4$ [✓ $t = 6$] $t = 12$