Math 115 — Final Exam — December 17, 2019

EXAM SOLUTIONS

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. Note that the back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
8. The use of any networked device while working on this exam is <u>not</u> permitted.
9. You may use any one calculator that does not have an internet or data connection except a TI-92 (or other calculator with a "qwerty" keypad). However, you must show work for any calculation which we have learned how to do in this course.
   You are also allowed two sides of a single $3'' \times 5''$ notecard.
10. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away.
14. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	6	
3	10	
4	7	
5	10	
6	8	

Problem	Points	Score
7	10	
8	9	
9	9	
10	7	
11	15	
Total	100	

1. [9 points] The function $f(x)$ is invertible and twice differentiable for all real numbers. The table to the right gives several values of $f'(x)$, **the derivative of** $f(x)$. You do not need to show work in this problem, but limited partial credit may be awarded for work shown.

x	-2	0	2	3	6
$f'(x)$	6	4	3	0	2

a. Compute each of the following values **exactly**. If there is not enough information, write NEI. If the value does not exist, write DNE.

i. [2 points] $\displaystyle\lim_{k\to 0}\frac{f(-2+k)-f(-2)}{k}$

Answer: 6

ii. [2 points] Let $h(x) = 3\cos(x)f(x)$. Find $h'(0)$.

Solution: $h'(x) = 3\cos(x)f'(x) - 3\sin(x)f(x)$

Answer: 12

iii. [2 points] Let $g(x) = f\left(\frac{6}{x}\right)$. Find $g'(3)$.

Solution: $g'(x) = f'\left(\frac{6}{x}\right)\left(\frac{-6}{x^2}\right)$

Answer: -2

b. [1 point] Use the table to give the best possible estimate of $f''(1)$.

Answer: $f''(1) \approx \dfrac{3-4}{2-0} = \dfrac{-1}{2}$

c. [2 points] Suppose that $f(6) = 0$. Write a formula for the linear approximation $L(x)$ of $f^{-1}(x)$, **the inverse of** $f(x)$, at $x = 0$.

Solution: The slope of $L(x)$ is $(f^{-1})'(0) = \dfrac{1}{f'(6)} = \dfrac{1}{2}$.

Answer: $L(x) = 6 + \frac{1}{2}x$

2. [6 points] Let $P(h)$ be the current pressure, in millibars (mb), of the air above Ann Arbor at a height of h meters (m) above the ground.

Use a complete sentence to write a practical interpretation of each of the following equations.

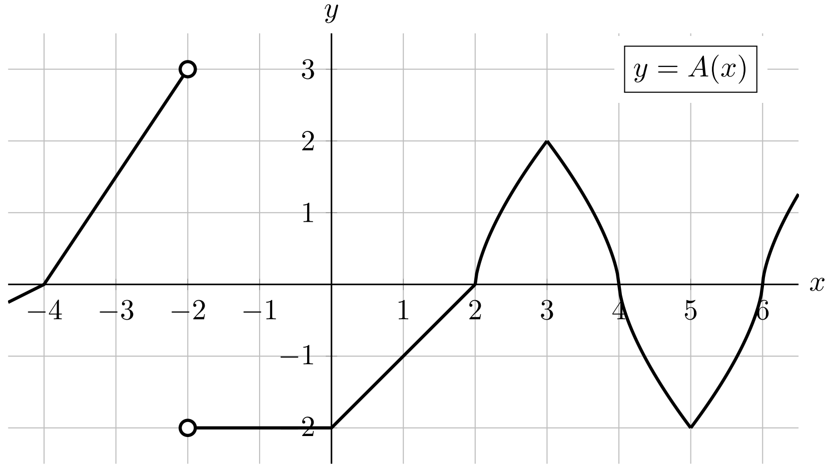
a. [3 points] $P'(6000) = -0.05$

Solution: The pressure at a height of 6100 meters above the ground is about 5 millibars lower than the pressure at 6000 meters above the ground.

b. [3 points] $\displaystyle\int_0^{4000} P'(h)\,dh = -510$

Solution: The pressure at a height of 4000 meters above the ground is 510 millibars lower than the pressure at ground level.

3. [10 points] A portion of the graph of a function $A(x)$ is shown below. Note that the part of the graph on the interval $[4, 6]$ can be obtained from the part of the graph on the interval $[2, 4]$ by shifting it two units to the right and reflecting it over the x-axis.



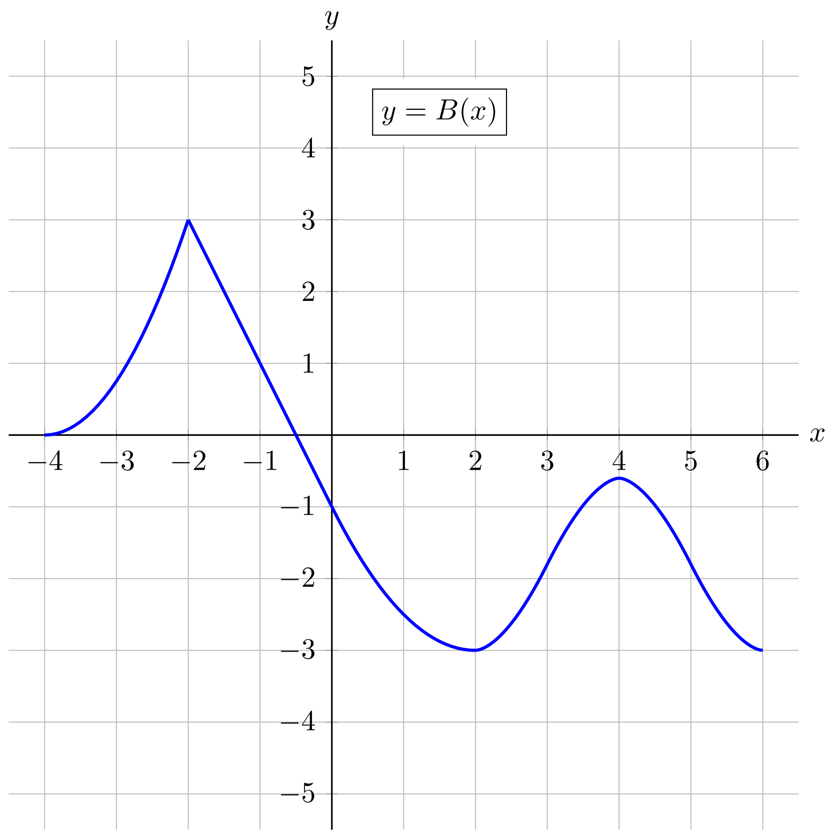
Let $B(x)$ be the continuous antiderivative of $A(x)$ passing through the point $(-1, 1)$.

a. [5 points] Use the graph above to complete the table below with the **exact** values of $B(x)$.

x	-4	-2	-1	0	2	6
$B(x)$	0	3	1	-1	-3	-3

b. [5 points] On the axes below, sketch a detailed graph of $y = B(x)$ for $-4 \le x \le 6$. Be sure that you pay close attention to each of the following:

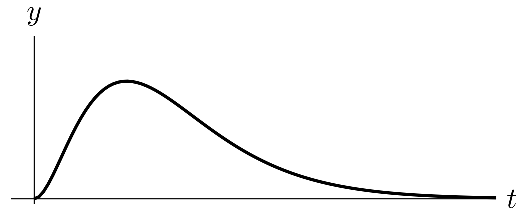
- where $B(x)$ is and is not differentiable,
- the values of $B(x)$ you found in the table above and at local extrema of B,
- where $B(x)$ is increasing/decreasing/constant, and the concavity of $B(x)$.



4. [7 points] Lin inflates a balloon using a helium pump. When she turns off the pump, the balloon immediately begins to deflate. Lin believes that she can model the balloon's volume, in cubic feet (ft^3), by the function

$$V(t) = \frac{at^2}{e^{bt}},$$

where t is the time, in seconds, after she begins inflating the balloon, and where a and b are positive constants. As an example, this function is shown to the right for one choice of the constants a and b. Note that the derivative of $V(t)$ is given by



$$V'(t) = -\frac{at(bt-2)}{e^{bt}}.$$

a. [4 points] The function $V(t)$ appears to have a local maximum at some time $t > 0$. Find the time at which this local maximum occurs. Use calculus to find your answer, and be sure to give enough evidence that the point you find is indeed a local maximum. Your answer may be in terms of a and/or b.

Solution: The critical points of $V(t)$ are 0 and $2/b$, and since b is positive, we know that $2/b > 0$. Below is a table showing the signs of $V'(t)$ for $t > 0$, with sign logic to justify how we know when $V'(t)$ is positive and when it is negative. Note: a and e^{bt} are always positive.

$V'(t)$	$(0, 2/b)$	$(2/b, +\infty)$
sign logic	$(-)\frac{(+)(-)}{(+)}$	$(-)\frac{(+)(+)}{(+)}$
sign	$(+)$	$(-)$

By the first derivative test, $t = 2/b$ is a local maximum. (The second derivative test can also be used.)

Answer: local max at $t =$ $2/b$

b. [3 points] Lin knows that it took 8 seconds to inflate the balloon, and that its volume at that time was 1.5 ft^3. Find the **exact** values of a and b for Lin's model. Show your work.

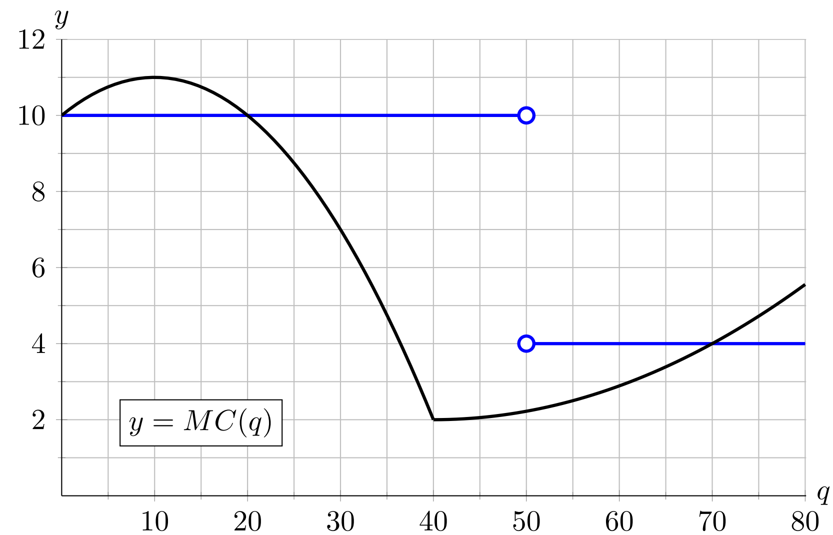
Solution: If Lin took 8 seconds to inflate the balloon, that means the local maximum we found in part (a) needs to occur at $t = 8$. If $2/b = 8$, then $b = 1/4$. The volume of the balloon at this time was 1.5 ft^3, which means

$$1.5 = V(8) = \frac{a(8)^2}{e^{(1/4)(8)}}.$$

Solving for a, we get $a = 1.5e^2/64$.

Answer: $a =$ $\dfrac{1.5e^2}{64}$ and $b =$ $1/4$

5. [10 points] Javier plans to make and sell his own all-natural shampoo. The graph below shows the marginal cost $MC(q)$, in dollars per liter, of q liters of shampoo. In order to start making shampoo, Javier must first spend \$25 on supplies, but he has no other fixed costs.



Javier can sell up to 50 liters of shampoo for \$10 per liter. Any additional shampoo can be sold to a local salon for \$4 per liter. Throughout this problem, you do not need to show work.

a. [2 points] On the axes above, carefully sketch the graph of the marginal revenue $MR(q)$, in dollars per liter, of q liters of shampoo.

Solution: See above.

b. [1 point] At what value(s) of q in the interval $[0, 80]$ is marginal cost maximized?

Answer: 10

c. [1 point] At what value(s) of q in the interval $[0, 80]$ is cost maximized?

Answer: 80

d. [2 points] At which values of q in the interval $[0, 80]$ is profit increasing? Give your answer as one or more intervals.

Answer: $(20, 70)$

e. [1 point] How many liters of shampoo should Javier make in order to maximize his profit?

Answer: 70

f. [3 points] Write an expression involving integrals which represents the company's profit when $q = 45$. Your expression may involve $MC(q)$ and/or $MR(q)$.

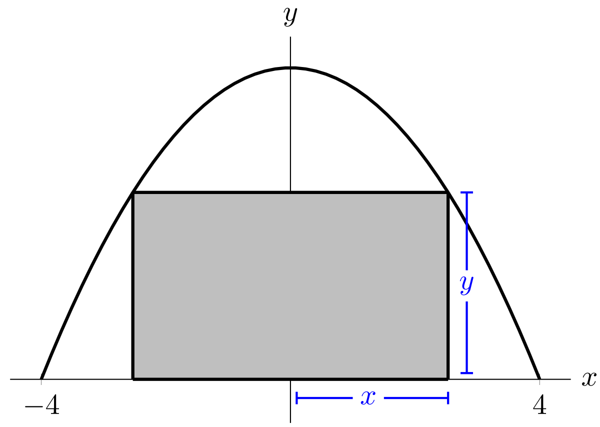
Answer: $-25 + \int_0^{45} (MR(q) - MC(q))dq$

6. [8 points]

A garden store plans to build a large rectangular sign on the interior wall at one end of their greenhouse. For x and y in meters, the curved roof of the greenhouse is described by the function

$$y = 5 - \frac{5}{16}x^2 \text{ for } -4 \le x \le 4.$$

This curve is graphed to the right; the shaded rectangle is one possible sign that could be built.



Find the width and height of the sign with the maximum area. Use calculus to find your answers, and be sure to show enough evidence that the values you find do in fact maximize the area.

Solution: Consider the rectangular sign labeled as in the above picture, with a width of $2x$ and height of y. The area of this sign is $A = (2x)(y)$. We also know that y can be expressed in terms of x, namely,

$$y = 5 - \frac{5}{16}x^2.$$

After making this substitution, the area of the sign is a function of x:

$$A(x) = (2x)\left(5 - \frac{5}{16}x^2\right) = 10x - \frac{5}{8}x^3.$$

The domain of this function is $[0, 4]$. To optimize, we need to find the critical points of $A(x)$ on its domain:

$$A'(x) = 10 - \frac{15}{8}x^2, \quad \text{so} \quad A'(x) = 0 \quad \text{when} \quad 10 = \frac{15}{8}x^2.$$

The critical points can be found by solving

$$\frac{15}{8}x^2 = 10 \quad \text{simplifies to} \quad x^2 = \frac{80}{15}, \quad \text{so there are two values:} \quad x = \pm\sqrt{\frac{80}{15}} \approx \pm 2.3094.$$

The value of $x \approx -2.3094$ is outside our domain of $[0, 4]$, so the only critical point we need to consider is $x \approx 2.3094$. We know that neither of endpoints of the interval $[0, 4]$ are the global maximum, since $A(0) = A(4) = 0$ but $A(2.3094) \approx 15.396 > 0$. Therefore, because there is only one critical point, it must be the global maximum.

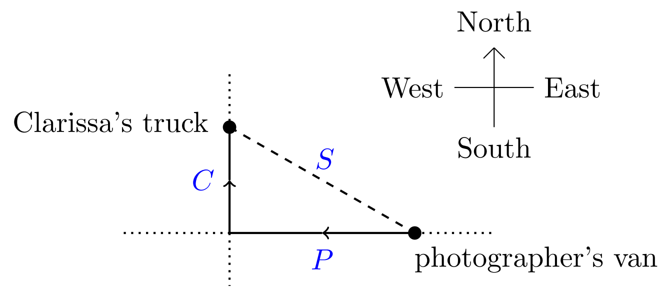
Now, the width of the sign is $2x$, not x, so the width that maximizes area is $2\sqrt{80/15} \approx 4.6188$. The corresponding height is

$$y = 5 - \frac{5}{16}\left(\sqrt{\frac{80}{15}}\right)^2 = \frac{10}{3}.$$

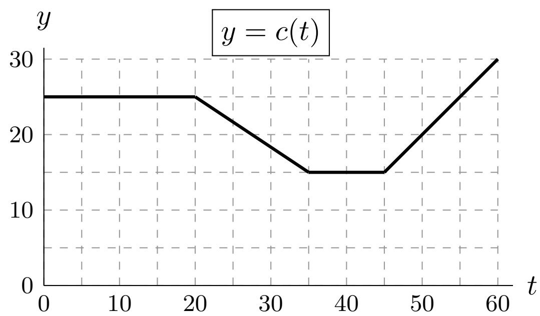
Answers: area is maximized when width = $2\sqrt{\frac{80}{15}} \approx 4.6188$ meters

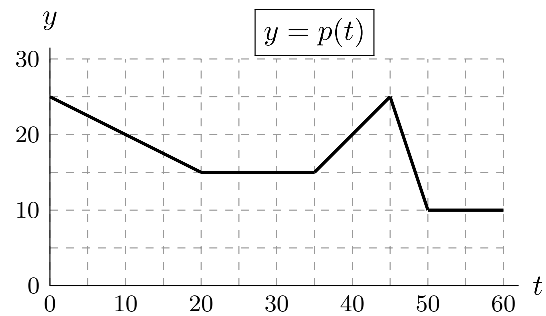
and height = $\frac{10}{3} \approx 3.3333$ meters

7. [10 points] Clarissa is driving a truck of showpigs, heading due north away from an intersection, while her photographer's van is due east of the intersection and driving due west, as shown to the right.



For time t measured in seconds after noon, let $c(t)$ be the speed of Clarissa's truck, and let $p(t)$ be the speed of the photographer's van, both measured in meters per second (m/s). Graphs of $c(t)$ and $p(t)$ are shown below.





At $t = 20$, Clarissa is 1000 meters from the intersection, and her photographer's van is 2400 meters from the intersection. Throughout this problem, be sure your work is clear.

a. [5 points] At $t = 20$, is the distance between the two vehicles increasing or decreasing? At what rate?

Solution: We can label the triangle as in the above picture. By the Pythagorean Theorem, $C^2 + P^2 = S^2$. Taking the derivative of both sides, we see that

$$2C\frac{dC}{dt} + 2P\frac{dP}{dt} = 2S\frac{dS}{dt}.$$

At time $t = 20$, we have $C = 1000$, $P = 2400$, and $S = \sqrt{C^2 + P^2} = 2600$. Since Clarissa is moving *away* from the intersection, $\frac{dC}{dt} = c(20) = 25$ and since the photographer is moving *towards* it, $\frac{dP}{dt} = -p(20) = -15$. Plugging in this information, we find that $\frac{dS}{dt} \approx -4.23077$.

Answer: INCREASING $\boxed{\text{DECREASING}}$ at a rate of: ≈ 4.23077 m/s

b. [2 points] How far from the intersection is the photographer's van at $t = 35$?

Solution:

$$P(35) = 2400 - \int_{20}^{35} p(t)dt = 2400 - 225 = 2175$$

Answer: 2175 meters

c. [3 points] What is the distance, in meters, between the two vehicles at $t = 35$?

Solution:

$$C(35) = 1000 + \int_{20}^{35} c(t)dt = 1000 + 300 = 1300$$

so using our answer from part (b), $S(35) = \sqrt{(2175)^2 + (1300)^2} \approx 2533.90$.

Answer: 2533.9 meters

8. [9 points] Given below is a table of values for a function $g(x)$ and its derivative $g'(x)$. The functions $g(x)$, $g'(x)$, and $g''(x)$ are all defined and continuous for all real numbers.

x	-3	-2	0	2	3	4	6	8
$g(x)$	2	3	7	9	5	1	-5	-7
$g'(x)$	0	4	1	0	-2	-4	-1	-3

Assume that between consecutive values of x given in the table above, $g(x)$ is either **always increasing** or **always decreasing**.

Find the quantities in **a.**–**c.** **exactly**, or write NEI if there is not enough information provided to do so. You do not need to show work, but limited partial credit may be awarded for work shown.

a. [1 point] $\displaystyle\int_3^6 g(x)\,dx$

Answer: **NEI**

b. [2 points] $\displaystyle\int_{-2}^{2} 3g'(x)\,dx$

Solution: $3(g(2) - g(-2)) = 3(9-3) = 18$

Answer: 18

c. [3 points] $\displaystyle\int_0^4 \left(g''(x) + x\right)\,dx$

Solution:

$$\int_0^4 \left(g''(x)+x\right)\,dx = \int_0^4 g''(x)\,dx + \int_0^4 x\,dx = (g'(4) - g'(0)) + \left(\frac{(4)^2}{2} - \frac{(0)^2}{2}\right)$$
$$= (-4-1) + 8$$

Answer: 3

d. [2 points] Use a right-hand Riemann sum with three equal subdivisions to estimate $\displaystyle\int_2^8 g(x)\,dx$. Write out all the terms in your sum.

Solution: $\Delta x = (8-2)/3 = 2$, so the Riemann sum is

$$g(4)\cdot 2 + g(6)\cdot 2 + g(8)\cdot 2 = 2\,(1 + (-5) + (-7)) = -22$$

e. [1 point] Does the answer to part **d.** overestimate, underestimate, or equal the value of $\displaystyle\int_2^8 g(x)\,dx$? Circle your answer. If there is not enough information, circle NEI.

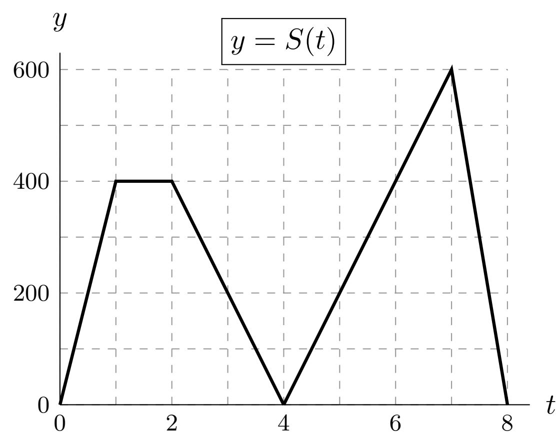
Answer: OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ EQUAL NEI

9. [9 points] Students from two rival universities had a competition to see who could clean up the most litter at a nature preserve.

University A went first, cleaning up litter from noon to 4pm. Each student from University A cleaned at a rate of 12 pounds of litter per hour.

Then University B cleaned up litter from 4pm to 8pm. Each student from University B cleaned at a rate of 9 pounds of litter per hour.

Let $S(t)$ be the number of students cleaning up litter at time t hours past noon. The graph of $S(t)$ is shown to the right.



a. [2 points] Find the total amount of litter cleaned up by University A. Show your work.

Solution:

$$12\int_0^4 S(t)\ dt = 12(1000)$$

Answer: 12000 pounds

b. [3 points] Find the total amount of litter cleaned up throughout the entire eight-hour competition. Show your work.

Solution:

$$12\int_0^4 S(t)dt + 9\int_4^8 S(t)\ dt = 12(1000) + 9(1200)$$

Answer: 22800 pounds

c. [4 points] The competition was broadcast live on TV. The number of people viewing the TV broadcast at time t hours past noon is given by the function

$$B(t) = 4S(t) + 200.$$

Find the average number of people viewing TV broadcast during the eight-hour competition.

Solution:

$$\begin{aligned}\frac{1}{8}\int_0^8 (4P(t)+200)\ dt &= \frac{4}{8}\int_0^8 S(t)\ dt + \frac{1}{8}\int_0^8 200\ dt \\ &= \frac{1}{2}(1000+1200) + \frac{1}{8}(200\cdot 8)\end{aligned}$$

Answer: 1300 people

10. [7 points] Consider the continuous function

$$f(x) = \begin{cases} -2 - \ln(x+2) & -2 < x \leq -1 \\ x2^{-x} & x > -1 \end{cases}$$

and its derivative

$$f'(x) = \begin{cases} -\dfrac{1}{x+2} & -2 < x < -1 \\ 2^{-x}(1 - x\ln(2)) & x > -1. \end{cases}$$

a. [2 points] Find all critical point(s) of $f(x)$. Write NONE if there are none.

Solution: We need to look for values of x where $f'(x)$ is zero or undefined. We can see that the first piece of $f'(x)$ is undefined at -2, but that's not in the domain, so it is not a critical point. We can also solve to see that the second piece of $f'(x)$ is zero when $x = \frac{1}{\ln(2)} \approx 1.443$, so this is in the domain and so a critical point.

We also can't forget that $f'(-1)$ may not be defined because there is where the pieces meet. In fact, the piecewise function given for $f'(x)$ isn't defined at $x = -1$. We can also check this by seeing that the two pieces of $f'(x)$ are not equal when we plug in $x = -1$. (The first piece is -1, while the second is $2(1 + \ln(2)) > 0$, so $f(x)$ has a corner here.)

Answer: critical point(s) at $x =$ $-1, \ \dfrac{1}{\ln(2)}$

b. [5 points] Find the x-coordinate of all global maxima and global minima of $f(x)$ on its domain $(-2, \infty)$. For each, write NONE if there are none. You must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

Solution: The critical points of $f(x)$ are $x = -1$, and $1/\ln(2)$ as we found above. We are optimizing on the domain $(-2, \infty)$, so aside from the critical points, we also need to consider the behavior at the endpoints $x = -2$ and the limit as x goes to ∞.

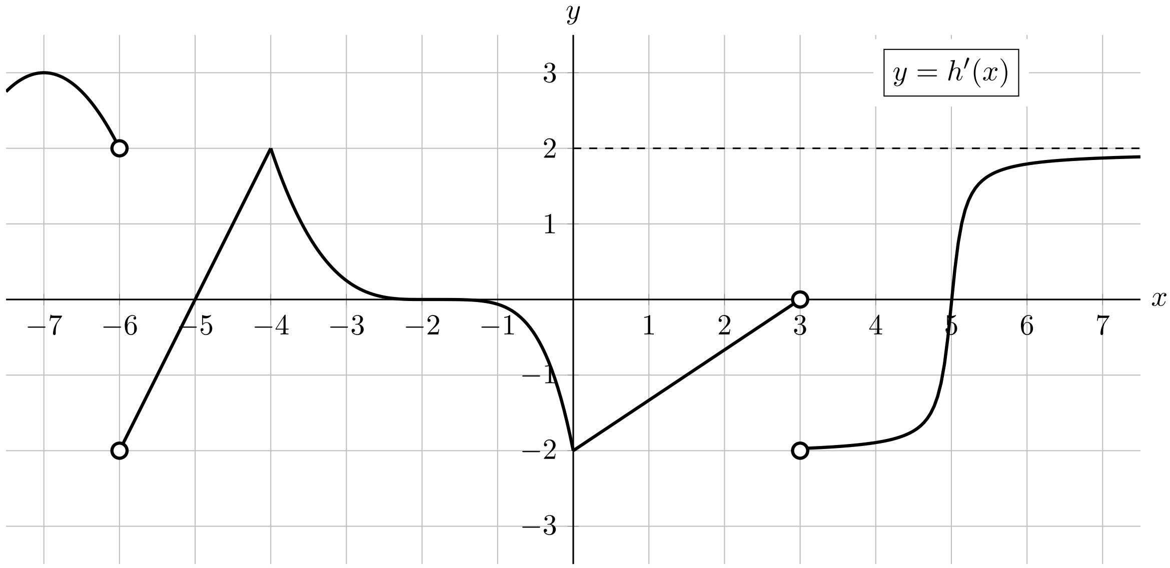
$\lim\limits_{x \to -2^+} f(x)$	$+\infty$
$f(-1)$	-2
$f(1/\ln(2))$	≈ 0.53074
$\lim\limits_{x \to \infty} f(x)$	0

The smallest output value is -2, so $x = -1$ is the global minimum. Since $\lim\limits_{x \to -2^+} f(x) = +\infty$, there is no global maximum

Answer: global max(es) at $x =$ **NONE**

Answer: global min(s) at $x =$ -1

11. [15 points] A function $h(x)$ is defined and continuous on $(-\infty, \infty)$. A portion of the graph of $h'(x)$, **the derivative of** $h(x)$, is shown below. Note that $y = 2$ is a horizontal asymptote of $y = h'(x)$.



In each part **a.–f.** below, circle all correct choices.

a. [2 points] At which of the following value(s) does $h(x)$ have a critical point?

$x = -7$ $\boxed{x = -5}$ $x = 0$ $\boxed{x = 3}$ NONE OF THESE

b. [2 points] At which of the following value(s) does $h(x)$ have a local maximum?

$\boxed{x = -6}$ $x = -4$ $\boxed{x = -2}$ $x = 5$ NONE OF THESE

c. [2 points] At which of the following value(s) does $h''(x)$ have a local maximum?

$x = -7$ $\boxed{x = -2}$ $\boxed{x = 5}$ $x = 6$ NONE OF THESE

d. [2 points] At which of the following value(s) does $h(x)$ have an inflection point?

$\boxed{x = -6}$ $x = -2$ $\boxed{x = 0}$ $x = 3$ NONE OF THESE

e. [2 points] On which of the following interval(s) is the average value of $h'(x)$ positive?

$\boxed{[-5, 0]}$ $\boxed{[-4, -2]}$ $[4, 5]$ NONE OF THESE

f. [2 points] On which of the following interval(s) is the average rate of change of $h'(x)$ positive?

$[-5, 0]$ $[-4, -2]$ $\boxed{[4, 5]}$ NONE OF THESE

g. [3 points] Find the following limits. If there is not enough information, write NEI. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

$\lim_{x \to \infty} h(x) =$ DNE $\qquad \lim_{x \to \infty} h'(x) =$ 2