# Math 115 — First Midterm — October 5, 2020

1. This exam has 7 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is not.

5. You must write your work and answers on **blank, white, physical paper**.
6. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
7. Make sure that all pages of work have the relevant problem number clearly identified.

| Problem | Points |
|---|---|
| 1 | 3 |
| 2 | 12 |
| 3 | 11 |
| 4 | 11 |
| 5 | 7 |

| Problem | Points |
|---|---|
| 6 | 11 |
| 7 | 12 |
| 8 | 11 |
| 9 | 8 |
| 10 | 9 |
| Total | 95 |

**1**. [3 points] **There** **<u>is</u>** **work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want handwritten (not typed) on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person other about the exam until **8am on Tuesday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

**As your submission for this problem, you must write "I agree," and write your initials and UMID number** to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

**2**. [12 points] Johannes takes a train from Buenos Aires to his countryside ranch, a distance of 1000 kilometers, traveling in a straight line. During the trip to the ranch, the train stops once, at the town of Rivadavia. After Johannes arrives at the ranch he realizes that he left an important book in Buenos Aires, so he returns to the city on an express train, which travels directly back to Buenos Aires on the same track with no stops.

Let $J(t)$ be Johannes's distance from Buenos Aires, in kilometers (km), at time $t$ hours (h) after the train begins moving. Some values of $J(t)$ are shown in the table below.

| $t$ | 0 | 3 | 7 | 8 | 9 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| $J(t)$ | 0 | 450 | 650 | 650 | 750 | 950 | 950 | 650 | 0 |

**a**. [2 points] How far is Rivadavia from Johannes's countryside ranch? Include units.

**b**. [2 points] What is the average velocity of the train between $t = 3$ and $t = 7$? Include units.

**c**. [2 points] Estimate the instantaneous velocity of the train 9 hours into Johannes's trip. Include units.

**d**. [2 points] For which of the following time intervals is the instantaneous velocity negative at some point in the interval? Give your answer as a list of one or more intervals, or write NONE.

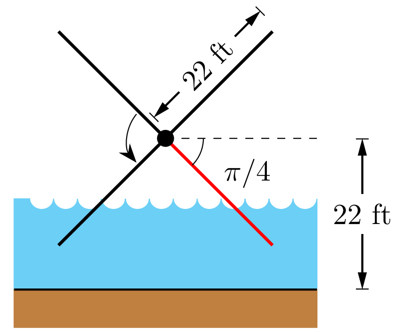
$[8, 12]$ $\qquad$ $[12, 14]$ $\qquad$ $[8, 16]$ $\qquad$ $[14, 18]$

**e**. [2 points] If the average velocity of the train on its return trip from the ranch to Buenos Aires was 200 km/h, and it arrived in Buenos Aires at exactly $t = 18$, at what time $t$ did the train depart?

**f**. [2 points] Could the graph of $J(t)$ be concave up for the entire interval $0 \le t \le 7$? *Briefly explain your reasoning.*

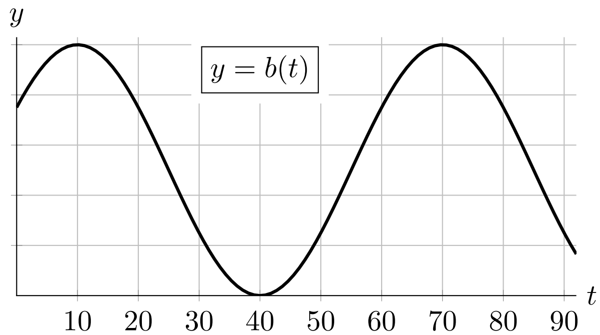
**3**. [11 points] You are standing by a river, watching three water wheels, each of which is rotating counterclockwise at a different but constant speed.

**a**. [4 points] The first water wheel takes 48 seconds to complete a full revolution. Each blade of the wheel is 22 feet long, and one of the blades is painted red. When each blade is at its lowest point, it just barely scrapes the bottom of the river. At the moment you begin watching, the red blade is exactly $\frac{\pi}{4}$ radians below the horizontal, as depicted to the right.




Write a formula for $r(t)$, the height, in feet, of the tip of the red blade above the bottom of the river $t$ seconds after you begin watching.

**b**. [4 points] Now you begin watching the second water wheel, which has one blade painted blue. Let $b(t)$ be the height, in feet, of the tip of the blue blade above the bottom of the river $t$ seconds after you begin watching. A portion of the graph of $b(t)$ is shown. Note that the scale on the $y$-axis is unknown.




The first time the blue blade reaches the water, since you began watching, is at $t = 28$.

i. At what time $t$ does the tip of the blue blade leave the water?

ii. At what time $t$ does the tip of the blue blade enter the water a second time?

**c**. [3 points] Finally, the third water wheel has a blade painted yellow, and you have determined that the height, in feet, of the tip of this blade above the bottom of the river $t$ seconds after you began watching is given by

$$40 + 35\sin(Bt),$$

where $B$ is some nonzero constant.

i. What is the length, in feet, of this yellow blade?

ii. How many feet above the bottom of the river is the center spoke of this water wheel?

**4**. [11 points]

**a**. [6 points] Consider the given table of values for the function $R(t)$.

| $t$ | 1 | 4 | 10 |
|---|---|---|---|
| $R(t)$ | 2 | 6 | 18 |

i. Could $R(t)$ be a linear function? Write YES or NO, show your work, and briefly explain your answer.

ii. Could $R(t)$ be an exponential function? Write YES or NO, show your work, and briefly explain your answer.

**b**. [5 points] Consider a different function $S(t)$, which is equal to 5 at $t = 0$, and decreases by 40% every 4 units of time. For which value of $t$ will $S(t)$ be equal to 1? *Show every step of your work, and give your final answer in exact form.*

**5**. [7 points]

**a**. [3 points] Let

$$Q(t) = 7 - \sin(t^2).$$

Suppose $k$ is a nonzero constant. Write an explicit expression for the average rate of change of $Q$ between $t = 5$ and $t = 5 + k$.

*Your answer should not involve the letter $Q$. Do not attempt to simplify your expression.*

Draw a box around your final answer.

**b**. [4 points] Let

$$P(w) = 6^{\arctan(4w)}.$$

Use the limit definition of the derivative to write an explicit expression for $P'(-3)$.

*Your answer should not involve the letter $P$. Do not attempt to evaluate or simplify the limit.*

Draw a box around your final answer.

**6**. [11 points] Define the following functions for an airplane taking off from a certain airport.

- Let $H(t)$ be the height above sea level, in kilometers (km), of the airplane $t$ minutes after takeoff.
- Let $T(k)$ be the temperature of the air outside the airplane, in degrees Celsius (°C), at a height of $k$ kilometers above sea level.

The functions $H(t)$ and $T(k)$ are differentiable and invertible.

**a**. [2 points] Use a complete sentence to give a practical interpretation of the equation $H^{-1}(6) = 5$.

**b**. [3 points] Write a single equation representing the following statement in terms of the functions $H, T$, and/or their inverses:

*The temperature of the air outside the airplane fell by 12 °C in the first five minutes after takeoff.*

**c**. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$T'(9) = -10.$$

*As the plane climbs from 8.8 km above sea level to 9 km above sea level...*

**d**. [3 points] Which of the following gives a valid interpretation of the equation $(H^{-1})'(4) = 0.5$? Write down the Roman numeral corresponding to your choice. There is only one correct answer.

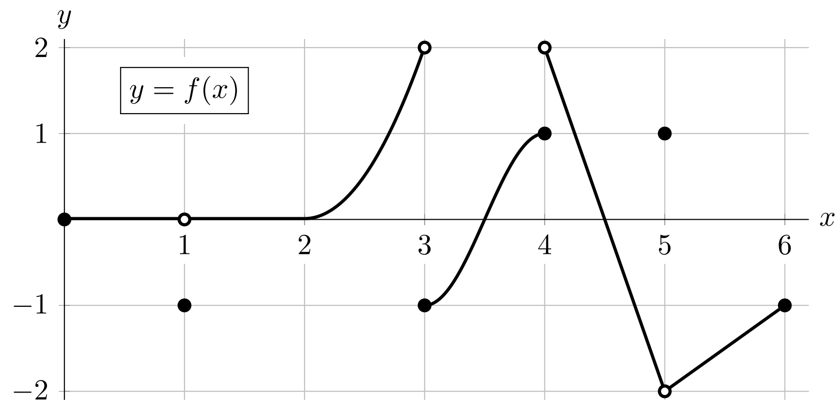
i. When the plane is at a height of 4 km, the temperature of the air outside the plane will decrease by about 0.5 degrees Celsius as the plane climbs an additional kilometer.

ii. It will take approximately 30 seconds for the airplane to climb from a height of 4 km to a height of 5 km.

iii. Four minutes into its flight, the plane will increase its height by about 0.5 km in the next minute.

iv. Once the plane has reached a height of 4 km, it will take about one minute to climb an additional 0.5 km.

**7**. [12 points] The graph of the function $f(x)$ is shown below.




For **a.–c.**, give your answers as a list of one or more of the given numbers, or write NONE.

**a**. [1 point] At which of the values $a = 1, 2, 3, 4, 5$ is $f(a)$ undefined?

**b**. [1 point] For which of the values $a = 1, 2, 3, 4, 5$ is $f(x)$ continuous at $x = a$?

**c**. [2 points] For which of the values $a = 1, 2, 3, 4, 5$ is $f(a) = \lim_{x \to a^-} f(x)$?

For **d.–g.**, use the graph of the function $f(x)$ to evaluate each of the expressions below. If a limit diverges to $\infty$ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

**d**. [2 points] $\lim_{x \to 5} f(x)$

**e**. [2 points] $\lim_{x \to 3} f(x)$

**f**. [2 points] $\lim_{x \to 0} f(4 + |x|)$

**g**. [2 points] $\lim_{h \to 0} \dfrac{f(4.25 + h) - f(4.25)}{h}$

**8**. [11 points] Consider the rational function $g(x) = \dfrac{(x-12)(x-7)(x-2)}{(2x-4)(x-3)(x-5)}$.

**a**. [2 points] What are the vertical asymptotes of the function $g(x)$?

**b**. [2 points] What are the vertical asymptotes of the function $\dfrac{1}{g(x)}$?

The piecewise function $h(x)$ is defined as follows, where $g(x)$ is as above, where $f(x)$ is from Problem 7 above, and where $B$ is a nonzero constant.

$$h(x) = \begin{cases} \dfrac{e^{2x}}{x^2} & x \le 3 \\ B \cdot f(x) & 3 < x \le 6 \\ g(x) & 6 < x \end{cases}$$

**c**. [3 points] Find an *exact* value of $B$ for which the function $h(x)$ is continuous at $x = 3$. Show your work.

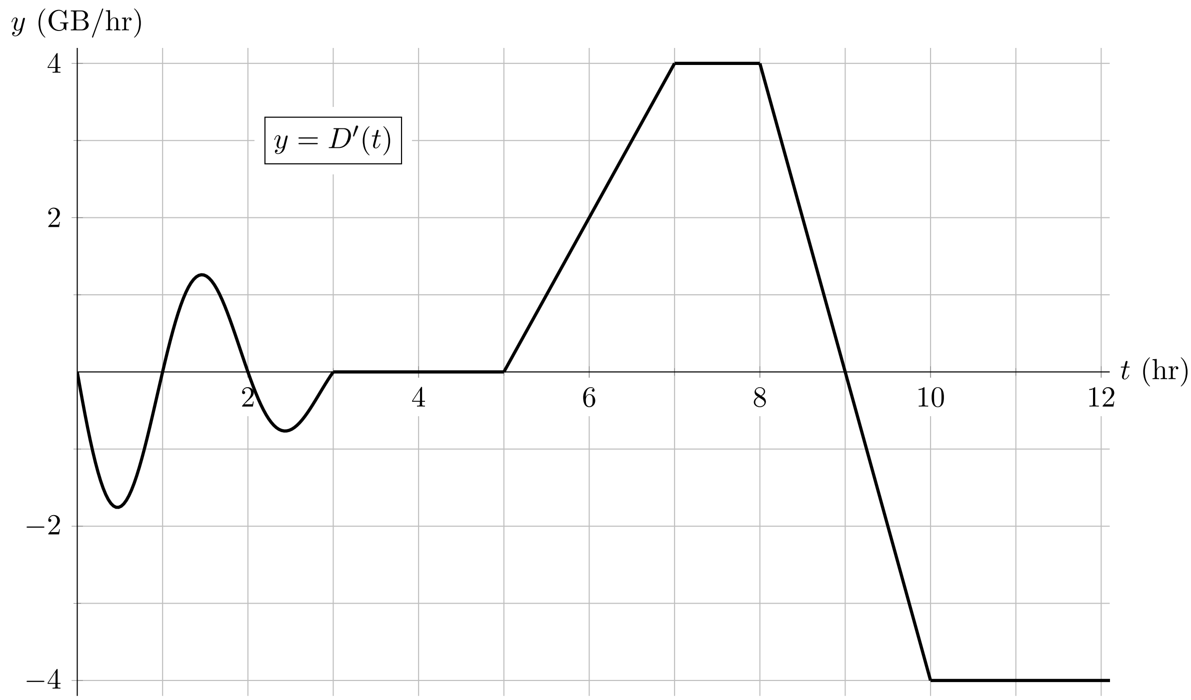
Evaluate each of the expressions below. If a limit diverges to $\infty$ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

**d**. [2 points] $\lim_{x \to \infty} h(x)$

**e**. [2 points] $\lim_{x \to -\infty} h(x)$

**9**. [8 points] The server for a website stores user data. Let $D(t)$ be the amount of user data stored on the server, in gigabytes (GB), at time $t$ hours after noon. Below is a portion of the graph of $D'(t)$, the **derivative** of $D(t)$. The function $D'(t)$ is

- constant for $3 \leq t \leq 5$, for $7 \leq t \leq 8$, and for $t \geq 10$, and is
- linear for $5 \leq t \leq 7$ and for $8 \leq t \leq 10$.




**a**. [2 points] On which of the following intervals of $t$ is the amount of user data stored on the server increasing for the entire interval? Give your answer as a list of one or more intervals, or write NONE.

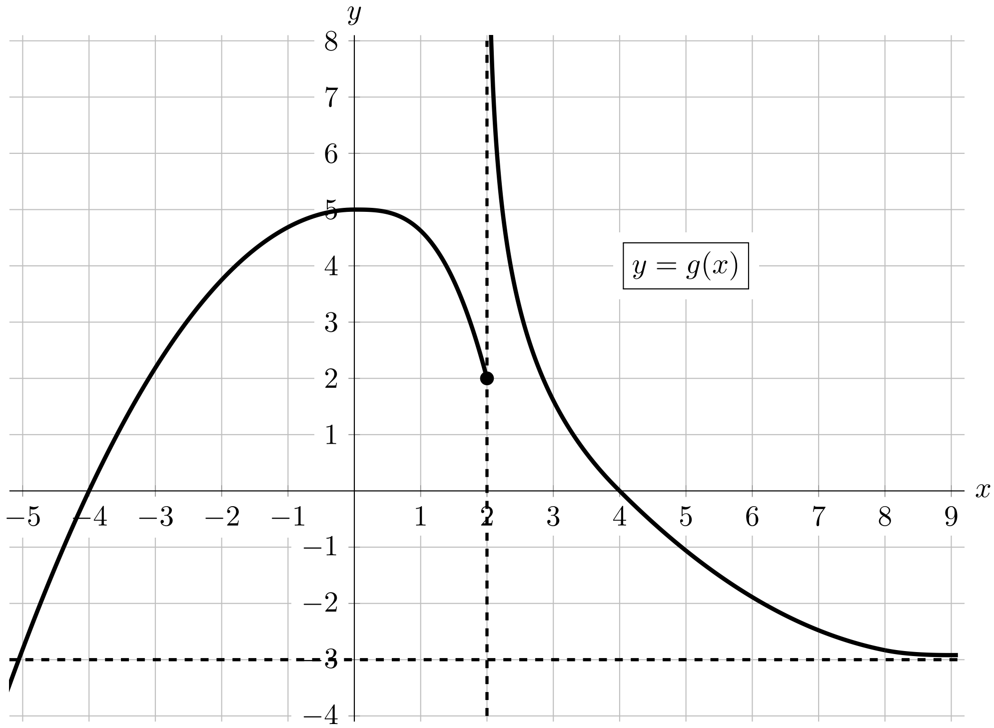
$(0.5, 1.5)$ $(1, 2)$ $(7, 8)$ $(10, 12)$

**b**. [2 points] When the amount of user data on the server is changing faster than 2 GB/hr, either increasing or decreasing, the server is said to be in an "excited state." How many hours, between noon and midnight, does the server spend in an excited state?

**c**. [2 points] The server hibernates when the amount of user data is not changing. How many hours, between noon and midnight, does the server spend in hibernation?

**d**. [2 points] At midnight, 450 GB of data is stored on the server. If the rate of change of user data stays the same from midnight to 5 am the following morning, how much user data will be stored on the server at 5 am?

**10.** [9 points] A portion of the graph of a function $g(x)$ is shown below.




The function $g$ has the following characteristics.

- A vertical asymptote at $x = 2$ (and no others).
- A horizontal asymptote at $y = -3$ (and no others).
- $g(x)$ is continuous and increasing on the interval $(-\infty, 0)$.
- $g(x)$ is continuous and decreasing on the interval $(2, \infty)$.
- The tangent line to the graph of $g(x)$ at $x = 0$ is horizontal.

**a.** [5 points] Consider $g'(x)$, the derivative of $g(x)$.
Determine whether each statement below is TRUE or FALSE. Write out the entire word TRUE or FALSE as your answer. No explanation is required.

i. $g'(-4) = 0$

ii. $g'(0) = 0$

iii. $g'(3) < g'(6)$

iv. $g'(-4) = g'(4)$

v. $g'(x)$ is decreasing on the interval $(-2, 1)$

**b.** [4 points] Consider the function $h(x) = 3g(x + 2)$.
Determine whether each statement below is TRUE or FALSE. Write out the entire word TRUE or FALSE as your answer. No explanation is required.

i. $h(x)$ is defined for all real numbers.

ii. The line $y = -1$ is a horizontal asymptote of the graph of $y = h(x)$.

iii. The line $x = 4$ is a vertical asymptote of the graph of $y = h(x)$.

iv. $h(x)$ is not continuous at $x = 0$.