Math 115 — First Midterm — October 5, 2020

EXAM SOLUTIONS

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1. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is not.
5. You must write your work and answers on **blank, white, physical paper**.
6. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
7. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points
1	3
2	12
3	11
4	11
5	7

Problem	Points
6	11
7	12
8	11
9	8
10	9
Total	95

1. [3 points] **There is work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5” by 11” standard sheet of paper, with whatever you want handwritten (not typed) on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person other about the exam until **8am on Tuesday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write “I agree,” and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

2. [12 points] Johannes takes a train from Buenos Aires to his countryside ranch, a distance of 1000 kilometers, traveling in a straight line. During the trip to the ranch, the train stops once, at the town of Rivadavia. After Johannes arrives at the ranch he realizes that he left an important book in Buenos Aires, so he returns to the city on an express train, which travels directly back to Buenos Aires on the same track with no stops.

Let $J(t)$ be Johannes's distance from Buenos Aires, in kilometers (km), at time t hours (h) after the train begins moving. Some values of $J(t)$ are shown in the table below.

t	0	3	7	8	9	12	14	16	18
$J(t)$	0	450	650	650	750	950	950	650	0

a. [2 points] How far is Rivadavia from Johannes's countryside ranch? Include units.

Solution: $1000 - 650 = 350$ km

(As stated in the problem, Johannes's ranch is 1000 km from Buenos Aires. The train only stops once on the way to the ranch, so the train must be stopped in Rivadavia between times $t = 7$ and $t = 8$. So Rivadavia is 650 km from Buenos Aires, and the distance between the ranch and Rivadavia is $1000 - 650 = 350$ km.)

b. [2 points] What is the average velocity of the train between $t = 3$ and $t = 7$? Include units.

Solution: $\frac{650-450}{7-3} = 50$ km/hr

c. [2 points] Estimate the instantaneous velocity of the train 9 hours into Johannes's trip. Include units.

Solution: There are several ways to reasonably estimate the instantaneous velocity at $t = 9$ using the table. One such estimate is given by the average velocity from $t = 8$ to $t = 9$. 9 hours into the trip, the instantaneous velocity is approximately $\frac{750-650}{9-8} = 100$ km/hr

d. [2 points] For which of the following time intervals is the instantaneous velocity negative at some point in the interval? Give your answer as a list of one or more intervals, or write NONE.

$[8, 12]$ $\boxed{[12, 14]}$ $\boxed{[8, 16]}$ $\boxed{[14, 18]}$

Solution: Some explanation (not necessary to earn full credit) follows.
Since the train turns around (reaches the ranch) between $t = 12$ and $t = 14$, the instantaneous velocity is never negative from $t = 0$ to $t = 12$ (including the interval $[8, 12]$) and is always negative from $t = 14$ to $t = 18$ (including the interval $[14, 18]$). It is also negative at some time in the interval $[12, 14]$, after the time when the train turns around.

e. [2 points] If the average velocity of the train on its return trip from the ranch to Buenos Aires was 200 km/h, and it arrived in Buenos Aires at exactly $t = 18$, at what time t did the train depart?

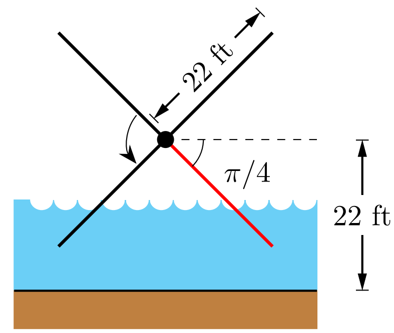
Solution: Let t be the time when the train left the ranch to return to Buenos Aires. On the return trip, the time elapsed is $18 - t$, the displacement is $1000 - 0$, and the average velocity is is 200 km/hr. So we have $\frac{1000-0}{18-t} = 200$ and find that the train left at time $t = 13$.

f. [2 points] Could the graph of $J(t)$ be concave up for the entire interval $0 \leq t \leq 7$? *Briefly explain your reasoning.*

Solution: The train's average velocity on the interval $0 \leq t \leq 3$ is $\frac{450-0}{3-0} = 150$ while its average velocity on the interval $3 \leq t \leq 7$ is $\frac{650-450}{7-3} = 50$. This means that the velocity cannot always be increasing, so the graph cannot be concave up for the entire interval. (Note that this conclusion agrees with our physical intuition: the train must slow down as it comes to a stop at the station in Rivadavia, which happens no later than time $t = 7$.)

3. [11 points] You are standing by a river, watching three water wheels, each of which is rotating counterclockwise at a different but constant speed.

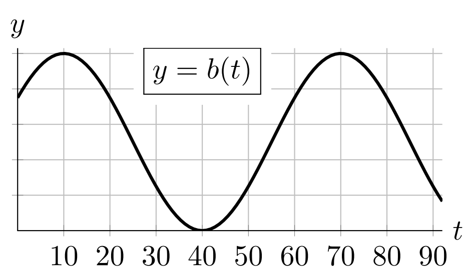
a. [4 points] The first water wheel takes 48 seconds to complete a full revolution. Each blade of the wheel is 22 feet long, and one of the blades is painted red. When each blade is at its lowest point, it just barely scrapes the bottom of the river. At the moment you begin watching, the red blade is exactly $\frac{\pi}{4}$ radians below the horizontal, as depicted to the right.



Write a formula for $r(t)$, the height, in feet, of the tip of the red blade above the bottom of the river t seconds after you begin watching.

Solution: The function $r(t)$ is sinusoidal.
Period: 48 seconds, Max: 44 ft, Min: 0 ft, Amplitude: 22 (the blades are 22 feet long)
Midline: $y = 22$ (the center of the wheel is 22 feet above the bottom of the river)
If the red blade were initially positioned at the horizontal instead of $\pi/4$ radians below it then the formula for $r(t)$ would be $22 + 22\sin\left(\frac{2\pi}{48}t\right)$.
However, the red blade is actually at the horizontal after 6 seconds (one eighth of a period), so one formula is $r(t) = 22 + 22\sin\left(\frac{2\pi}{48}(t-6)\right) = 22 + 22\sin\left(\dfrac{\pi}{24}t - \dfrac{\pi}{4}\right)$.
Other equivalent answers include: $r(t) = 22 + 22\cos\left(\frac{2\pi}{48}(t-18)\right) = 22 + 22\cos\left(\frac{\pi}{24}t - \frac{3\pi}{4}\right)$ and $r(t) = 22 - 22\cos\left(\frac{2\pi}{48}(t+6)\right) = 22 - 22\cos\left(\frac{\pi}{24}t + \frac{\pi}{4}\right)$.

b. [4 points] Now you begin watching the second water wheel, which has one blade painted blue. Let $b(t)$ be the height, in feet, of the tip of the blue blade above the bottom of the river t seconds after you begin watching. A portion of the graph of $b(t)$ is shown. Note that the scale on the y-axis is unknown.



The first time the blue blade reaches the water, since you began watching, is at $t = 28$.

i. At what time t does the tip of the blue blade leave the water?

ii. At what time t does the tip of the blue blade enter the water a second time?

Solution: i. Let w be the time when the blade leaves the water, a number larger than 40. Since the graph is symmetric about the line $t = 40$, the distance from w to 40 is the same as the distance from 40 to 28, the time when the blade enters the water. In other words, the blade reaches the bottom of the wheel halfway between when it enters and exits the water.

$$w - 40 = 40 - 28 \quad \text{so} \quad w = 52$$

Alternatively, since the graph peaks at $t = 10$ and $t = 70$, the distance from 70 to w is the same as the distance from 28 to 10: $70 - w = 28 - 10$, so again, $w = 52$.
ii. Since the first peak of the function is at $t = 10$ and the second peak of the function is at $t = 70$, the period of the function is $70 - 10 = 60$. The tip of the blue blade enters the water again after one full period has elapsed, at time $t = 28 + 60 = 88$.

c. [3 points] Finally, the third water wheel has a blade painted yellow, and you have determined that the height, in feet, of the tip of this blade above the bottom of the river t seconds after you began watching is given by

$$40 + 35\sin(Bt),$$

where B is some nonzero constant.

i. What is the length, in feet, of this yellow blade?

ii. How many feet above the bottom of the river is the center of this water wheel?

Solution:
i. The length of the yellow blade is the amplitude of this sinusoidal function, 35.
ii. The distance from the bottom of the river to the center of the wheel is the midline of the sinusoidal function, 40.

4. [11 points]

a. [6 points] Consider the given table of values for the function $R(t)$.

t	1	4	10
$R(t)$	2	6	18

i. Could $R(t)$ be a linear function? Write YES or NO, show your work, and briefly explain your answer.

Solution: If $R(t)$ were linear then its average rate of change over any interval would be the same. However, the average rate of change on $[1, 4]$ is $\frac{6-2}{4-1} = \frac{4}{3}$ whereas the average rate of change on $[4, 10]$ is $\frac{18-6}{10-4} = \frac{12}{6} = 2$. Since these rates of change are not equal, $R(t)$ cannot be linear.

ii. Could $R(t)$ be an exponential function? Write YES or NO, show your work, and briefly explain your answer.

Solution: If $R(t)$ were exponential then its growth factor over any interval would be constant. On the one hand, the growth factor from 1 to 4 would be the solution to the equation $a^{4-1} = \frac{6}{2} = 3$, that is, $a = 3^{1/3}$. On the other hand, the growth factor from 4 to 10 would be the solution to the equation $a^{10-4} = \frac{18}{6} = 3$, that is, $a = 3^{1/6}$. Since these growth factors are not the same, $R(t)$ cannot be exponential.

b. [5 points] Consider a different function $S(t)$, which is equal to 5 at $t = 0$, and decreases by 40% every 4 units of time. For which value of t will $S(t)$ be equal to 1? *Show every step of your work, and give your final answer in exact form.*

Solution: The growth factor a is the solution to the equation $a^4 = 1 - 0.4$, that is, $a = (0.6)^{1/4}$. Since the initial value is $S(0) = 5$, a formula for $S(t)$ is $S(t) = 5\big((0.6)^{1/4}\big)^t = 5(0.6)^{t/4}$. Now we solve for t in the equation $S(t) = 1$.

$$\begin{aligned} 5(0.6)^{t/4} &= 1 \\ 0.6^{t/4} &= 0.2 \\ \ln(0.6^{t/4}) &= \ln(0.2) \\ \frac{t}{4} \cdot \ln(0.6) &= \ln(0.2) \\ t &= \frac{4\ln(0.2)}{\ln(0.6)} \end{aligned}$$

5. [7 points]

a. [3 points] Let

$$Q(t) = 7 - \sin(t^2).$$

Suppose k is a nonzero constant. Write an explicit expression for the average rate of change of Q between $t = 5$ and $t = 5 + k$.

Your answer should not involve the letter Q. Do not attempt to simplify your expression.

Draw a box around your final answer.

Solution:

$$\boxed{\frac{7 - \sin\left((5+k)^2\right) - \left(7 - \sin(5^2)\right)}{k}}$$

b. [4 points] Let

$$P(w) = 6^{\arctan(4w)}.$$

Use the limit definition of the derivative to write an explicit expression for $P'(-3)$.

Your answer should not involve the letter P. Do not attempt to evaluate or simplify the limit.

Draw a box around your final answer.

Solution:

$$\boxed{\lim_{h \to 0} \frac{6^{\arctan(4(-3+h))} - 6^{\arctan(4(-3))}}{h}}$$

6. [11 points] Define the following functions for an airplane taking off from a certain airport.

- Let $H(t)$ be the height above sea level, in kilometers (km), of the airplane t minutes after takeoff.
- Let $T(k)$ be the temperature of the air outside the airplane, in degrees Celsius (°C), at a height of k kilometers above sea level.

The functions $H(t)$ and $T(k)$ are differentiable and invertible.

a. [2 points] Use a complete sentence to give a practical interpretation of the equation $H^{-1}(6) = 5$.

Solution: The airplane takes 5 minutes to reach a height of 6 kilometers.

b. [3 points] Write a single equation representing the following statement in terms of the functions H, T, and/or their inverses:

The temperature of the air outside the airplane fell by 12 °C in the first five minutes after takeoff.

Solution:

$$T(H(5)) - T(H(0)) = -12$$

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$T'(9) = -10.$$

As the plane climbs from 8.8 km above sea level to 9 km above sea level...

Solution: "the temperature drops by about 2 °C."

d. [3 points] Which of the following gives a valid interpretation of the equation $(H^{-1})'(4) = 0.5$? Write down the Roman numeral corresponding to your choice. There is only one correct answer.

i. When the plane is at a height of 4 km, the temperature of the air outside the plane will decrease by about 0.5 degrees Celsius as the plane climbs an additional kilometer.

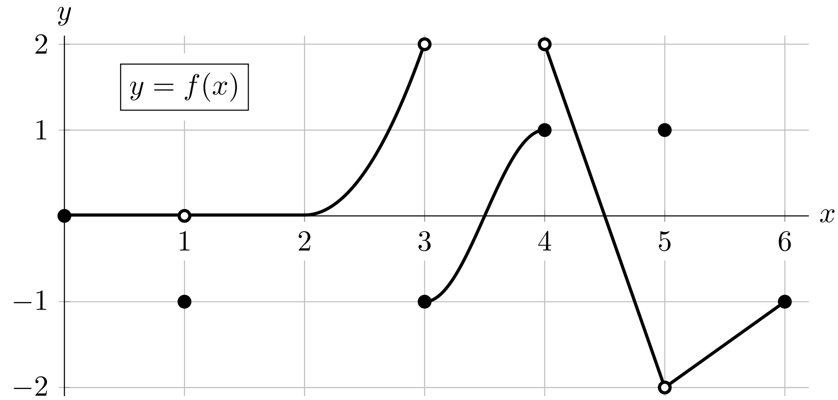
$\boxed{\text{ii.}}$ It will take approximately 30 seconds for the airplane to climb from a height of 4 km to a height of 5 km.

iii. Four minutes into its flight, the plane will increase its height by about 0.5 km in the next minute.

iv. Once the plane has reached a height of 4 km, it will take about one minute to climb an additional 0.5 km.

Solution: The correct answer is ii.

7. [12 points] The graph of the function $f(x)$ is shown below.



For **a.–c.**, give your answers as a list of one or more of the given numbers, or write NONE.

a. [1 point] At which of the values $a = 1, 2, 3, 4, 5$ is $f(a)$ undefined?

Solution: NONE

b. [1 point] For which of the values $a = 1, 2, 3, 4, 5$ is $f(x)$ continuous at $x = a$?

Solution: $a = 2$

c. [2 points] For which of the values $a = 1, 2, 3, 4, 5$ is $f(a) = \lim_{x \to a^-} f(x)$?

Solution: $a = 2$, 4

For **d.–g.**, use the graph of the function $f(x)$ to evaluate each of the expressions below. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

d. [2 points] $\lim_{x \to 5} f(x)$

Solution: Since $\lim_{x \to 5^-} f(x) = -2$ and $\lim_{x \to 5^+} f(x) = -2$, the two-sided limit is -2.

e. [2 points] $\lim_{x \to 3} f(x)$

Solution: Since $\lim_{x \to 3^-} f(x) = 2$ while $\lim_{x \to 3^+} f(x) = -1$, the two-sided limit does not exist.

f. [2 points] $\lim_{x \to 0} f(4 + |x|)$

Solution: As x approaches 0 from both sides, $|x|$ approaces 0 through positive values. So $\lim_{x \to 0} f(4 + |x|) = \lim_{t \to 4^+} f(t) = 2$.

g. [2 points] $\lim_{h \to 0} \dfrac{f(4.25 + h) - f(4.25)}{h}$

Solution: This expression represents the slope of the tangent line to the graph of $g(x)$ at $x = 4.25$. On the interval $(4, 5)$ the graph is linear with slope -4; so the limit is -4.

8. [11 points] Consider the rational function $g(x) = \dfrac{(x-12)(x-7)(x-2)}{(2x-4)(x-3)(x-5)}$.

a. [2 points] What are the vertical asymptotes of the function $g(x)$?

Solution: The vertical asymptotes of $g(x)$ are $x = 3$ and $x = 5$.
(Note that the numerator of $g(x)$ is equal to 0 when $x = 12$, $x = 7$, and $x = 2$, while the denominator is equal to 0 when $x = 2$, $x = 3$, and $x = 5$. Now $g(x) = \frac{(x-12)(x-7)}{2(x-3)(x-5)}$ when $x \neq 2$ so $\lim\limits_{x\to 2} g(x)$ exists. Therefore $g(x)$ has a "hole" at $x = 2$ rather than a vertical asymptote.)

b. [2 points] What are the vertical asymptotes of the function $\dfrac{1}{g(x)}$?

Solution: The vertical asymptotes of $\dfrac{1}{g(x)}$ are $x = 7$ and $x = 12$.
(The function $\frac{1}{g(x)}$ is obtained from the function $g(x)$ by swapping its numerator and denominator. Therefore, the vertical asymptotes of $\frac{1}{g(x)}$ are the zeroes of $g(x)$, namely, $x = 7$ and $x = 12$.)

The piecewise function $h(x)$ is defined as follows, where $g(x)$ is as above, where $f(x)$ is from Problem 7 above, and where B is a nonzero constant.

$$h(x) = \begin{cases} \dfrac{e^{2x}}{x^2} & x \leq 3 \\ B \cdot f(x) & 3 < x \leq 6 \\ g(x) & 6 < x \end{cases}$$

c. [3 points] Find an *exact* value of B for which the function $h(x)$ is continuous at $x = 3$. Show your work.

Solution: Note that $h(3) = \lim\limits_{x\to 3^-} h(x) = \dfrac{e^6}{9}$. while $\lim\limits_{x\to 3^+} B \cdot f(x) = B \cdot f(3) = -B$. In order for $f(x)$ to be continuous at $x = 3$, these limits must be equal to each other, so $B = -\dfrac{e^6}{9}$.

Evaluate each of the expressions below. If a limit diverges to ∞ or $-\infty$ or if the limit does not exist for any other reason, write DNE.

d. [2 points] $\lim\limits_{x\to\infty} h(x)$

Solution: Since $h(x) = g(x)$ when $x > 6$, this limit is the same as $\lim\limits_{x\to\infty} g(x)$. The limit of this rational function is determined by the leading terms in the numerator and denominator. The leading term of the numerator is x^3, and the leading term of the denominator is $2x^3$. So

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x^3}{2x^3} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}.$$

e. [2 points] $\lim\limits_{x\to-\infty} h(x)$

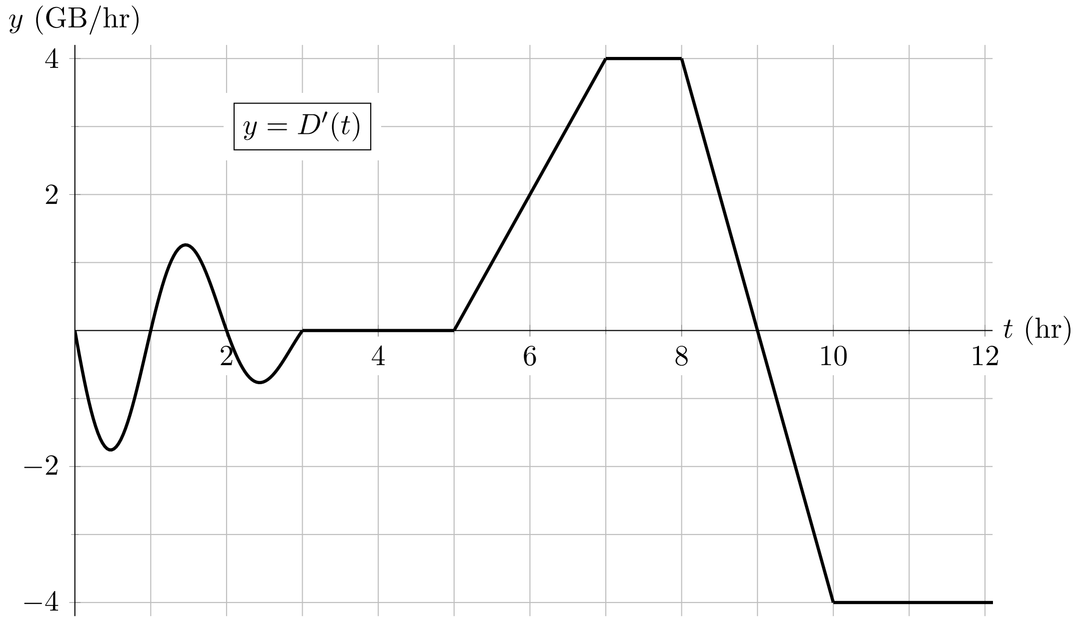
Solution:

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{e^{2x}}{x^2} = 0.$$

(The first equality holds since $h(x) = \frac{e^{2x}}{x^2}$ when $x \leq 3$. As x grows without bound in the negative direction, e^{2x} approaches 0 while x^2 grows without bound. So $\lim\limits_{x\to-\infty} \frac{e^{2x}}{x^2} = 0$.)

9. [8 points] The server for a website stores user data. Let $D(t)$ be the amount of user data stored on the server, in gigabytes (GB), at time t hours after noon. Below is a portion of the graph of $D'(t)$, the **derivative** of $D(t)$. The function $D'(t)$ is

- constant for $3 \le t \le 5$, for $7 \le t \le 8$, and for $t \ge 10$, and is
- linear for $5 \le t \le 7$ and for $8 \le t \le 10$.



a. [2 points] On which of the following intervals of t is the amount of user data stored on the server increasing for the entire interval? Give your answer as a list of one or more intervals, or write NONE.

$(0.5, 1.5)$ $\boxed{(1,2)}$ $\boxed{(7,8)}$ $(10, 12)$

Solution: (The amount of user data is increasing when its derivative, the function $D'(t)$ graphed above, is positive.)

b. [2 points] When the amount of user data on the server is changing faster than 2 GB/hr, either increasing or decreasing, the server is said to be in an "excited state." How many hours, between noon and midnight, does the server spend in an excited state?

Solution: The server spends $\boxed{\text{5 hours}}$ in an excited state.
(This question is asking for the times t when $D'(t) > +2$ or $D'(t) < -2$. The graph lies above the line $y = +2$ for $6 < t < 8.5$and lies below the line $y = -2$ for $6 < t < 8.5$.)

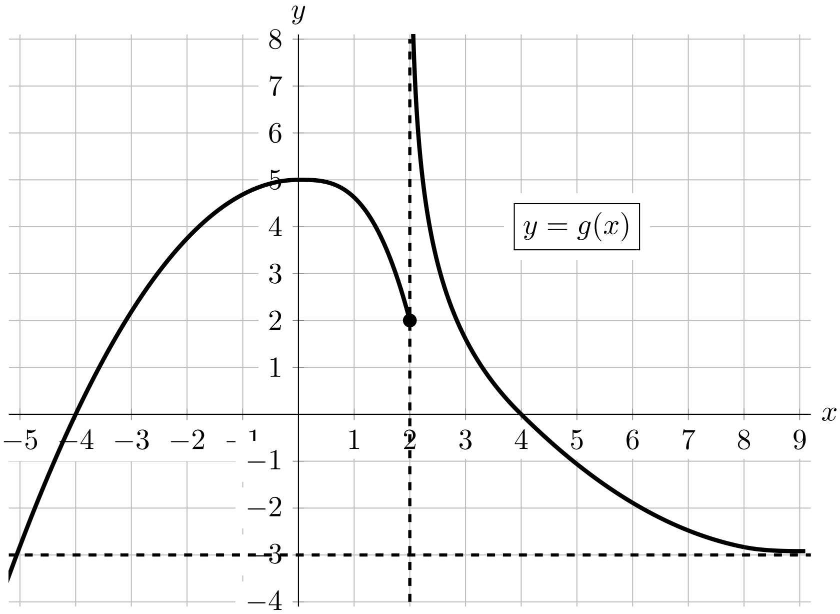
c. [2 points] The server hibernates when the amount of user data is not changing. How many hours, between noon and midnight, does the server spend in hibernation?

Solution: The amount of user data $D(t)$ is not changing exactly when the derivative $D'(t)$ equals zero. The total time in hibernation is therefore $5 - 3 = \boxed{\text{2 hours}}$.

d. [2 points] At midnight, 450 GB of data is stored on the server. If the rate of change of user data stays the same from midnight to 5 am the following morning, how much user data will be stored on the server at 5 am?

Solution: The graph above shows that $D'(12) = -4$, so at midnight, the amount of data is changing at a rate of -4 GB/hr. From midnight to 5 am the total change in the amount of data is therefore $5 \text{ hr} \cdot (-4 \text{ GB/hr}) = -20$ GB. Since the server has 450 GB of data at midnight, there must be $450 - 20 = \boxed{\text{430 GB}}$ of data at 5 am.

10. [9 points] A portion of the graph of a function $g(x)$ is shown below.



The function g has the following characteristics.

- A vertical asymptote at $x = 2$ (and no others).
- A horizontal asymptote at $y = -3$ (and no others).
- $g(x)$ is continuous and increasing on the interval $(-\infty, 0)$.
- $g(x)$ is continuous and decreasing on the interval $(2, \infty)$.
- The tangent line to the graph of $g(x)$ at $x = 0$ is horizontal.

a. [5 points] Consider $g'(x)$, the derivative of $g(x)$.
Determine whether each statement below is TRUE or FALSE. Write out the entire word TRUE or FALSE as your answer. No explanation is required.

i. $g'(-4) = 0$ FALSE

ii. $g'(0) = 0$ TRUE

iii. $g'(3) < g'(6)$ TRUE

iv. $g'(-4) = g'(4)$ FALSE

v. $g'(x)$ is decreasing on the interval $(-2, 1)$ TRUE

Solution: Remember, $g'(a)$ is the slope of the tangent line to the graph of $g(x)$ at $x = a$.

b. [4 points] Consider the function $h(x) = 3g(x + 2)$.
Determine whether each statement below is TRUE or FALSE. Write out the entire word TRUE or FALSE as your answer. No explanation is required.

i. $h(x)$ is defined for all real numbers. TRUE

ii. The line $y = -1$ is a horizontal asymptote of the graph of $y = h(x)$. FALSE

iii. The line $x = 4$ is a vertical asymptote of the graph of $y = h(x)$. FALSE

iv. $h(x)$ is not continuous at $x = 0$. TRUE

Solution: Note that the graph of $h(x)$ is obtained from the graph of $g(x)$ by first shifting the graph to the left by 2 units and then scaling (stretching) it vertically by a factor of 3.