Math 115 — Second Midterm — November 9, 2020

1. This exam has 13 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is <u>not</u>.
5. You must write your work and answers on **blank, white, physical paper**.
6. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
7. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points
1	3
2	5
3	6
4	13
5	10
6	14

Problem	Points
7	5
8	10
9	9
10	9
11	6
12	10
Total	100

1. [3 points] **There <u>is</u> work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want handwritten (not typed) on both sides.
- You are <u>not</u> allowed to use any other resources, including calculators, other notes, or the book.
- You may <u>not</u> use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may <u>not</u> use help from any other individuals (other students, tutors, online help forums, etc.), and may <u>not</u> communicate with any other person about the exam until **8am on Tuesday, November 10** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

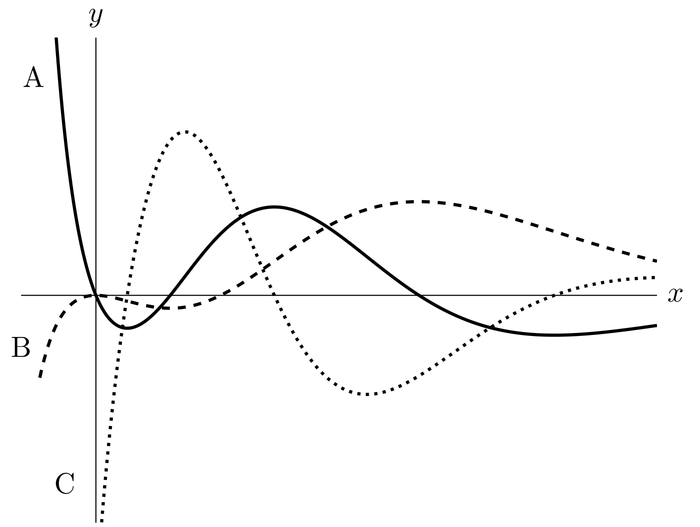
2. [5 points] Shown on the axes below are portions of the graphs of $y = f(x)$, $y = f'(x)$, and $y = f''(x)$.

Determine which graph is which.
Write your answer as a list of the form

$f(x)$:
$f'(x)$:
$f''(x)$:

indicating after each function the letter A, B, or C that corresponds to its graph.

No work or justification is needed.



Solution:

$f(x)$: B
$f'(x)$: A
$f''(x)$: C

3. [6 points] The function $g(x)$ is given by the equation

$$g(x) = \begin{cases} ax^2 & x \leq 1 \\ b - \ln(3x) & x > 1 \end{cases}$$

where a and b are constants. Find one pair of **exact** values for a and b such that $g(x)$ is differentiable, or write NONE if there are none. Be sure your work is clear.

Solution:
Continuity at $x = 1$ requires:

$$a(1^2) = b - \ln(3 \cdot 1)$$
$$a = b - \ln(3).$$

So $b = a + \ln(3)$.

Note that $\frac{d}{dx}(ax^2) = 2ax$ and $\frac{d}{dx}(b - \ln(3x)) = -\frac{1}{x}$
So differentiability at $x = 1$ also requires:

$$2a(1) = -\frac{1}{1}$$
$$2a = -1$$
$$a = -\frac{1}{2}.$$

Therefore, $a = -\dfrac{1}{2}$ and $b = -\dfrac{1}{2} + \ln 3$.

4. [13 points] Suppose $p(x)$ is a continuous function defined for all real numbers x. The **derivative** and **second derivative** of $p(x)$ are given by

$$p'(x) = |x|\,(x+4)^3 \qquad \text{and} \qquad p''(x) = \frac{4x(x+1)(x+4)^2}{|x|}.$$

Throughout this problem, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [1 point] Find the x-coordinates of all critical points of $p(x)$. If there are none, write NONE.

Solution: $x = -4,\ 0.$

b. [2 points] Find the x-coordinates of all critical points of $p'(x)$. If there are none, write NONE.

Solution: $x = -4,\ -1,\ 0.$

c. [5 points] Find the x-coordinates of

i. all local minima of $p(x)$ and

ii. all local maxima of $p(x)$.

If there are none of a particular type, write NONE.

Solution: The second derivative test is inconclusive, so we have to use the first derivative test. Here's a number line showing a sign-logic calculation for the test:

$(+)(-) = (-)$ | -4 | $(+)(+) = (+)$ | 0 | $(+)(+) = (+)$ → $p'(x)$

(We could also compute $p'(x)$ at values of x between critical points.) We can conclude that $p(x)$ has a local minimum at $x = -4$ and no local maximum.

d. [5 points] Find the x-coordinates of all inflection points of $p(x)$. If there are none, write NONE.

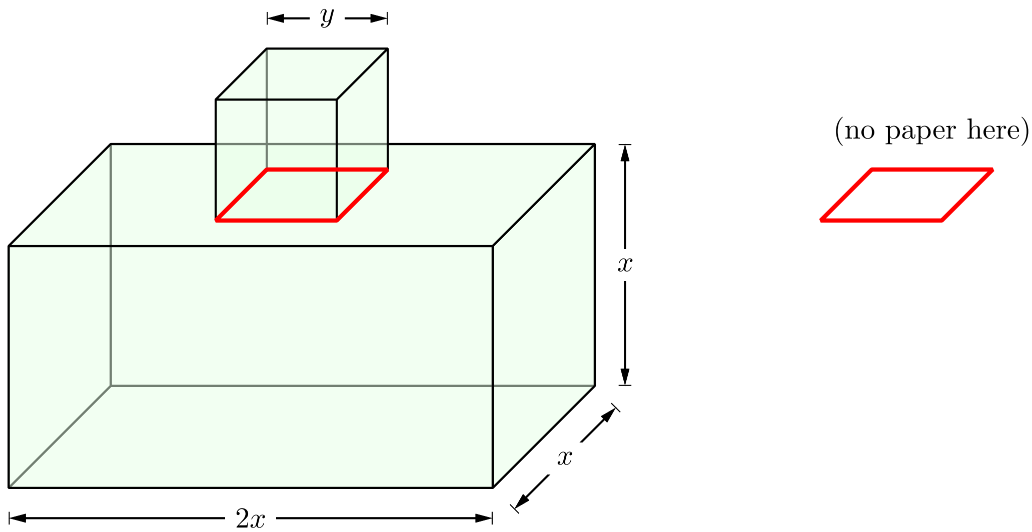
Solution: We know from part bthat the candidate points are $-4, -1$, and 0, so we test whether $p(x)$ changes concavity at these points by finding the signs of $p''(x)$:

$\frac{(-)(-)(+)}{(+)} = (+)$ | -4 | $\frac{(-)(-)(+)}{(+)} = (+)$ | -1 | $\frac{(-)(+)(+)}{(+)} = (-)$ | 0 | $\frac{(+)(+)(+)}{(+)} = (+)$ → $p''(x)$

So $p(x)$ has an inflection point at $x = -1$ and $x = 0$.

5. [10 points] An architect is building a model out of wire and paper.

- The lower part is a box of length $2x$ centimeters (cm), depth x cm, and height x cm.
- The top part is a cube of side length y cm.
- The top part is attached to the lower part at the center of the top of the lower part.
- The architect requires that $0 \le y \le x$.
- Paper will cover the outside of the model: there is paper on the sides of the upper and lower parts, including the bottom, but no paper where the upper and lower parts meet.



The architect will use exactly 160 cm^2 of paper to make the model.

a. [4 points] Write a formula for y in terms of x.

Solution: The surface area of the lower part is $4 \cdot 2x^2 + 2 \cdot x^2 = 10x^2$, but we should subtract y^2 to account for the hole in the top face. The surface area of the sides and top (but without the bottom) of the top part is $5y^2$. So the total amount of paper needed is $10x^2 - y^2 + 5y^2 = 10x^2 + 4y^2 = 160$. So $y = \sqrt{40 - \frac{10}{4}x^2}$.

b. [2 points] Write a formula for the function $V(x)$ which gives the total volume of the model in terms of x only.

Solution: The total volume is $2x^3 + y^3 = 2x^3 + \left(\sqrt{40 - \frac{10}{4}x^2}\right)^3$.

c. [4 points] In the context of this problem, what is the domain of $V(x)$?

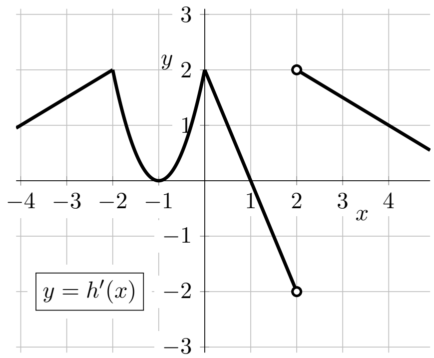
Solution:
The value of x would be largest when $y = 0$, in which case $10x^2 + 4(0)^2 = 160$, so that $x = 4$.

The smallest x can be is y, so that $10x^2 + 4x^2 = 160$ and $x = \sqrt{160/14}$.

So the domain is $\sqrt{\frac{160}{14}} \le x \le 4$.

6. [14 points] A table of values for a differentiable function $g(x)$ and its derivative $g'(x)$ are shown below to the left. Below to the right is shown a portion of the graph of $h'(x)$, the <u>derivative</u> of a function $h(x)$. The function $h(x)$ is defined and continuous for all real numbers.

x	-1	0	1	3	4
$g(x)$	0	2	5	1	-7
$g'(x)$	4	3	-1	-6	-3



Answer parts **a.–b.**, or write NONE if appropriate. You do not need to show work.

a. [2 points] List the x-coordinates of all critical points of $h(x)$ on the interval $(-4, 4)$.

Solution: $x = -1,\ 1,\ 2$

b. [2 points] List the x-coordinates of all local maxima of $h(x)$ on the interval $(-4, 4)$.

Solution: $x = 1$

Find the **exact** values for parts **c.–e.**, or NEI if there is not enough information to do so. Write DNE if the value does not exist. Your answers should not include the letters g or h but you do not need to simplify. Show work.

c. [2 points] Let $A(x) = \dfrac{\sin(x) + 3}{g(x)}$. Find $A'(0)$.

Solution: $A'(x) = \dfrac{g(x)\cos(x) - (\sin(x) + 3)g'(x)}{g(x)^2}$, so $A'(0) = \dfrac{2 \cdot 1 - (0 + 3) \cdot 3}{2^2} = -\dfrac{7}{4}$.

d. [2 points] Let $f(x) = g(h'(x))$. Find $f'(4)$.

Solution: $f'(x) = h''(x)g'(h'(x))$, so $f'(4) = -\dfrac{1}{2} \cdot g'(1) = \dfrac{1}{2}$.

e. [2 points] Let $P(x) = xe^{g(x)}$. Find $P'(-1)$.

Solution: $P'(x) = e^{g(x)} + xg'(x)e^{g(x)}$, so $P'(-1) = e^0 + (-1)(4)e^0 = -3$.

Answer parts **f.–g.** You do not need to show work.

f. [2 points] Complete the following sentence.

Because the function $g(x)$ satisfies the hypotheses of the mean value theorem on the interval $[-1, 4]$, there must be some point c with $-1 \le c \le 4$ such that...

Solution: $g'(c) = -\dfrac{7}{5}$.

g. [2 points] On which of the following intervals does $h'(x)$ satisfy the hypotheses of the mean value theorem? List all correct answers, or write NONE.

$[-2, 0]$ $\qquad$ $[-1, 1]$ $\qquad$ $[3, 4]$

Solution: $[-2, 0]$ and $[3, 4]$

7. [5 points] An implicit function is described by the equation

$$\cos(xy) = 7x^2 + y.$$

Find a formula for $\dfrac{dy}{dx}$ in terms of y and x. You must show every step of your work.

Solution:

$$-\sin(xy)\left(y + x\,\frac{dy}{dx}\right) = 14x + \frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - \frac{dy}{dx} = 14x + y\sin(xy)$$

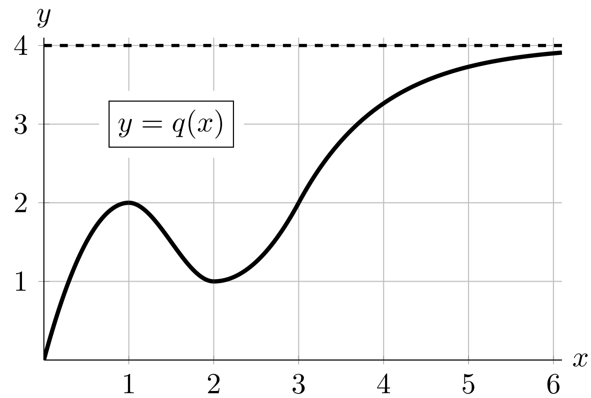
$$-\frac{dy}{dx}(x\sin(xy) + 1) = 14x + y\sin(xy)$$

$$\frac{dy}{dx} = -\frac{14x + y\sin(xy)}{x\sin(xy) + 1}$$

8. [10 points]

The graph of a function $q(x)$ is shown on the right. Note that:

- the domain of $q(x)$ is $[0, \infty)$,
- $q(x)$ has critical points at $x = 1$ and $x = 2$,
- $q(x)$ has no critical points for $x \geq 4$, and
- $q(x)$ has a horizontal asymptote at $y = 4$.



Now consider the piecewise-defined function $r(x)$ given as follows, where $q(x)$ is as given above:

$$r(x) = \begin{cases} \frac{1}{2}x^3 - \frac{3}{2}x & \text{if } x \leq 0 \\ q(x) & \text{if } x > 0 \end{cases}.$$

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

a. [5 points] Find the x-coordinates of

i. the global minimum(s) of $r(x)$ on $[-1, 3]$ and

ii. the global maximum(s) of $r(x)$ on $[-1, 3]$.

If there are none of a particular type, write NONE.

Solution: The derivative of $\frac{1}{2}x^3 - \frac{3}{2}x$ is

$$\frac{3}{2}x^2 - \frac{3}{2} = \frac{3}{2}(x-1)(x+1),$$

so $r(x)$ has critical points at $x = -1$ for the left piece, and 1 and 2 from the right piece. Since $0^3 - 3(0) = 0 = q(0)$, the function r is continuous and we can use the extreme value theorem. But, note that $x = 0$ is a critical point because there is a corner there, since the slope from the left approaches $-\frac{3}{2}$, while the slope on the right side is clearly positive. To determine which of our critical points and endpoints might be global extrema, we make a table of values.

x	-1	0	1	2	3
$r(x)$	1	0	2	1	2

Hence a global minimum occurs at $x = 0$ and global maxima occur at $x = 1$ and 3.

b. [5 points] Find the x-coordinates of

i. the global minimum(s) of $r(x)$ on $(-\infty, \infty)$ and

ii. the global maximum(s) of $r(x)$ on $(-\infty, \infty)$.

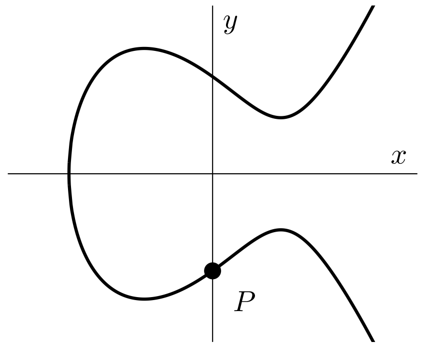
If there are none of a particular type, write NONE.

Solution: In the previous part we examined all the critical points of $r(x)$, so in this part we need only examine the end behavior. Since

$$\lim_{x\to-\infty} r(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} r(x) = 4,$$

there are no global maxima or minima on this interval: both answers are NONE.

9. [9 points] Let $\mathcal{C}$ be the curve given by the equation $y^2 + 3x = x^3 + 3$. The graph of $\mathcal{C}$ is shown below.



Note that $\dfrac{dy}{dx} = \dfrac{3x^2 - 3}{2y}$. You must show all of your work in this problem.

a. [2 points] Find the coordinates of the point P.

Solution: $y^2 + 3(0) = 0^3 + 3$, so $(0, -\sqrt{3})$

b. [3 points] The point $(-2, 1)$ is on the curve $\mathcal{C}$. Find the equation of the tangent line to the curve $\mathcal{C}$ at this point.

Solution: Since the slope at this point is $\dfrac{dy}{dx} = \dfrac{3(-2)^2 - 3}{2(1)} = \dfrac{9}{2}$, the tangent line is $y = 1 + \dfrac{9}{2}(x + 2)$.

c. [4 points] Find all points on the curve $\mathcal{C}$ where the tangent line is horizontal. Give your answer as a list of ordered pairs. Write NONE if there are no such points.

Solution: To find where the tangent line is horizontal, set $3x^2 - 3 = 0$ to find $x = \pm 1$.

When $x = 1$, we find $y^2 + 3(1) = (1)^3 + 3$, or $y^2 = 1$, so $y = \pm 1$.
This leads to the points $(1, 1)$ and $(1, -1)$.

When $x = -1$, we find $y^2 + 3(-1) = (-1)^3 + 3$, or $y^2 = 5$, so $y = \pm\sqrt{5}$.
This leads to the points $(-1, \sqrt{5})$, and $(-1, -\sqrt{5})$.

10. [9 points] A box is to be constructed with a square base. The height of the box and the side length of the square base must add up to 3 meters (m).

- Find the height and side length of the square base, in m, that lead to a box of maximum volume.
- What is the maximum volume in this case, in cubic meters?

In your solution, make sure to carefully define any variables and functions you use, use calculus to justify your answers, and show enough evidence that the values you find do in fact maximize the volume.

Solution:
Let h be the height and x the side length of the square base. We must have $h + x = 3$.

Then the total volume is

$$V(x) = x^2 h = x^2(3 - x).$$

We want to maximize $V(x)$ on the domain $0 \leq x \leq 3$. Since the derivative is

$$V'(x) = 6x - 3x^2 = 3x(2 - x),$$

the critical points are $x = 0$ and $x = 2$.

Note that the volume is zero when $x = 0$ or $x = 3$, but positive for $x = 2$. Or, we could show that $x = 2$ is a local maximum and note that, since it's the only critical point on (the interior of) our domain $[0, 3]$, it must be the global max.

Thus the maximum occurs when $x = 2$ cm and $h = 1$ cm, then the maximum volume is $2^2(3-1) = 4$.

11. [6 points] Suppose that $C = h(T)$ is the daily cost, in dollars, to heat a certain house if the average outside temperature that day is T degrees Fahrenheit (°F). The function $h(T)$ is invertible and differentiable.

a. [3 points] Complete the following sentence to give a practical interpretation of $h'(40) = -0.1$.

If one day the average outside temperature is 40°F *and the next day it is* 37°F, . . .

Solution: "then the daily cost to heat the house will have increased by about 0.3 dollars."

b. [3 points] Complete the following sentence to give a practical interpretation of $(h^{-1})'(3.6) = -8$.

If the cost to heat the house increased from \$3.60 *on one day to* \$3.70 *the next day,* . . .

Solution: "then the temperature must have decreased by approximately 0.8°F."

12. [10 points] Again suppose that $C = h(T)$ is the daily cost, in dollars, to heat a certain house if the average outside temperature that day is T degrees Fahrenheit (°F). Some values of $h(T)$ and its derivative $h'(T)$ are given in the table below.

T	5	8	18	30	55
$h(T)$	8	7.2	5	3.3	1.4
$h'(T)$	−0.3	−0.25	−0.2	−0.11	−0.05

The function $h(T)$ is invertible and differentiable. Also, $h''(T)$ exists and is positive for all T.

a. [2 points] Find the linear approximation $L(T)$ of $h(T)$ near $T = 8$.

Solution: $L(T) = 7.2 - 0.25(T - 8)$

b. [1 point] Use your formula for $L(T)$ to approximate $h(10)$.

Solution: $h(10) \approx 7.2 - 0.25 \cdot 2 = 6.7$

c. [2 points] Is your answer in part **b.** an overestimate or underestimate of the actual value, or is there not enough information to decide? Briefly explain.

Solution: The approximation is an underestimate since $h''(T)$ is positive, so $h(T)$ is concave up.

d. [3 points] Suppose that the quadratic approximation $Q(T)$ of $h(T)$ near $T = 25$ is given by

$$Q(T) = 3.9 - 0.15\,(T - 25) + 0.003\,(T - 25)^2.$$

Find the values of $h(25)$, $h'(25)$, and $h''(25)$.

Solution:
$h(25) = 3.9$
$h'(25) = -0.15$
$h''(25) = 0.003 * 2 = 0.006$

e. [2 points] Use the table to compute $(h^{-1})'(5)$.

Solution: $(h^{-1})'(5) = \dfrac{1}{h'(18)} = -5$