Math 115 — First Midterm — October 4, 2021

EXAM SOLUTIONS

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1. You may use both sides of a 3"x5" notecard, but you may **not use any other resources, including calculators, other notes, the book, or other individuals (other students, tutors, online help forums, etc.).**

2. This exam has 8 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
4. You must use the methods learned in this course to solve all problems.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.33333$ is not.

7. You must write your work and answers on **blank, white, physical paper**.
8. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
9. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points	Score
1	10	
2	5	
3	4	
4	9	

Problem	Points	Score
5	11	
6	7	
7	10	
8	4	
Total	60	

1. [10 points] The dogs Elvis and Judy are playing with their owner.

At time $t = 0$ seconds, Elvis starts to run in a straight line away from his owner. After running 110 feet, Elvis turns around and runs back to his owner along the same straight line. Some values of the differentiable function $E(t)$, which gives Elvis's distance from his owner, in feet, at time t seconds, are recorded in the table below. Assume that Elvis's velocity is positive when he is moving away from his owner.

t	2	5	7	10	12	15	17	18
$E(t)$	33	78	98	98	74	44	34	10

a. [2 points] What was Elvis's average velocity between times $t = 12$ and $t = 17$? Include units.

Solution:

$$\frac{E(17) - E(12)}{17 - 12} = \frac{34 - 74}{5} = -8$$

Answer: -8 **ft/sec**

b. [2 points] Estimate Elvis's instantaneous velocity at $t = 5$. Include units.

Solution:

$$E'(5) \approx \frac{E(7) - E(5)}{7 - 5} = \frac{98 - 78}{2} = 10$$

Answer: 10 **ft/sec**

c. [2 points] For which of the following time interval(s) must $E'(t) = 0$ for some value of t in that time interval? Give your answer as a list of one or more of these intervals, or write NONE.

$(2, 5)$ $(6, 9)$ $\boxed{(7, 10)}$ $(15, 17)$

Judy is also running along a straight line, sometimes away from and sometimes back toward her owner. The function $J(t)$ gives Judy's distance from her owner, in feet, at time t seconds. Some values of $J'(t)$, the **derivative** of $J(t)$, are given in the table below. Assume that $J'(t)$ is continuous, decreasing on the interval $(4, 11)$, and increasing on the interval $(11, 22)$.

t	4	7	8	11	15	17	20	22
$J'(t)$	10	2	0	-12	-3	0	1	9

d. [2 points] On which of the following time interval(s) is Judy running away from her owner? Give your answer as a list of one or more of these intervals, or write NONE.

$\boxed{(4, 7)}$ $(8, 11)$ $(11, 15)$ $\boxed{(20, 22)}$

e. [2 points] At which of the following times is Judy's speed the greatest? Write the one correct answer.

$t = 4$ $\boxed{t = 11}$ $t = 17$ $t = 20$

2. [5 points] Let

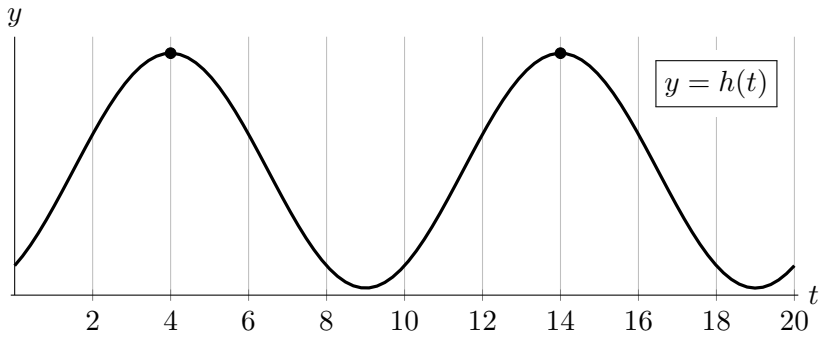
$$B(w) = w + e^{3w}.$$

Use the limit definition of the derivative to write an explicit expression for $B'(-1)$. *Your answer should not involve the letter B. Do not attempt to evaluate or simplify the limit.*
Draw a box around your final answer.

Answer: $B'(-1) =$ $\lim_{h\to 0} \dfrac{\left((-1+h)+e^{3(-1+h)}\right)-\left(-1+e^{3(-1)}\right)}{h}$

3. [4 points] A ride at an amusement park straps riders into a seat that travels vertically up and down a tower. The park has added a camera to a nearby building to take pictures of riders when the seat is at the same height as the camera. The graph of the sinusoidal function $h(t)$ below gives the height of the seat t seconds after the ride begins. The scale and units of the vertical axis are unknown, but the park has determined that the first time the seat is at the same height as the camera is when t is exactly 1.95 seconds.

Find the **exact** values of the next two times t when the seat is at the same height as the camera. You do not need to show work, but limited partial credit may be awarded for work shown. Make sure it is clear what your final answers are.



Solution: Maxima at $t = 4$ and $t = 14$, so the period is 10 seconds.
Time from camera height to maximum height $= 4 - 1.95 = 2.05$ seconds.
Time from maximum height to next camera height $= 2.05$ seconds, so next time at camera height is $4 + 2.05 = 6.05$ seconds. The second time is one full period after the original.

Answer: $t =$ **6.05**

Answer: $t =$ **11.95**

4. [9 points]
Consider the piecewise function

$$q(x) = \begin{cases} 7e^{x-C} + \dfrac{x}{3x-2} & x < 0 \\ \dfrac{6+5x}{2+3x+4^x} & x \geq 0 \end{cases}$$

where C is a constant.

a. [5 points] Find an exact value of C for which the function $q(x)$ is continuous at $x = 0$, or write NONE if there is no such value. Show your work and make sure your final answer is clear.

Solution: Set both pieces of the function equal for $x = 0$ and solve for C:

$$7e^{0-C} + \frac{0}{3(0)-2} = \frac{6+5(0)}{2+3(0)+4^0}$$
$$7e^{-C} = 2$$

(Below is one possible approach to solving for C.)

$$\ln(7e^{-C}) = \ln(2)$$
$$\ln(7) + \ln(e^{-C}) = \ln(2)$$
$$\ln(7) - C = \ln(2)$$
$$C = \ln(7) - \ln(2)$$

Answer: $C =$ $\ln(7) - \ln(2)$

Evaluate each of the expressions below. If a limit does not exist, including if it diverges to $\pm\infty$, write DNE. You do not need to show work.

b. [2 points] $\lim_{x\to\infty} q(x)$

Solution:

$$\lim_{x\to\infty} \frac{6+5x}{2+3x+4^x} = 0$$

Answer: 0

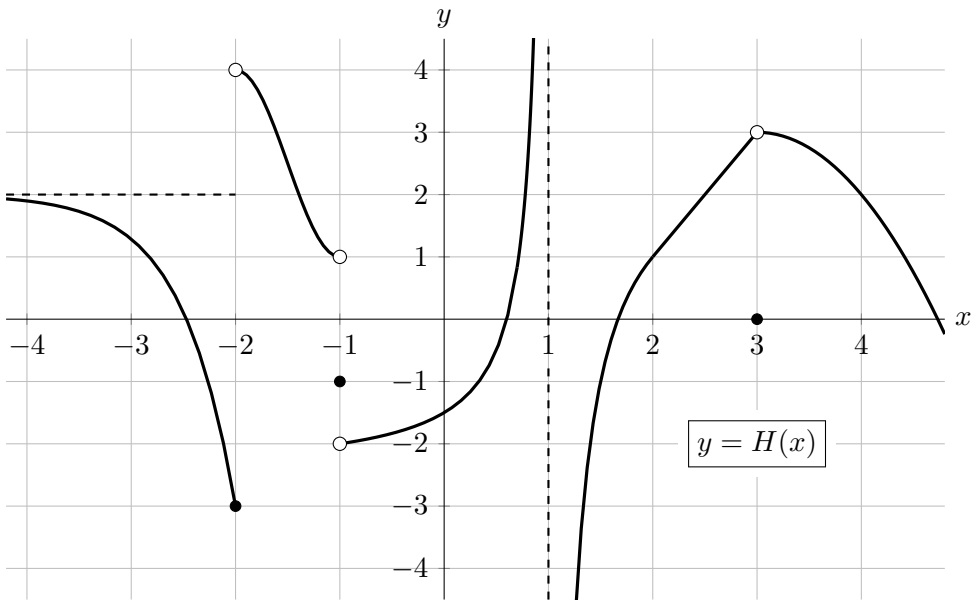
c. [2 points] $\lim_{x\to-\infty} q(x)$

Solution:

$$\lim_{x\to-\infty} \left(7e^{x-C} + \frac{x}{3x-2}\right) = \frac{1}{3}$$

Answer: 1/3

5. [11 points] Below is a portion of the graph of the function $H(x)$, which is defined for all $x < 1$ and all $x > 1$. Note that $H(x)$ is linear for $2 \le x < 3$, and that $H(x)$ has a horizontal asymptote at $y = 2$ and a vertical asymptote at $x = 1$.



Use this graph to find the numerical value of each of the following limits. If a limit does not exist, including if it diverges to $\pm\infty$, write DNE. You do not need to show work.

a. [1 point] $\lim_{x \to -2} H(x)$

Answer: DNE

b. [1 point] $\lim_{x \to 3} H(x)$

Answer: 3

c. [1 point] $\lim_{x \to -\infty} H(x)$

Answer: 2

d. [2 points] $\lim_{x \to 4^+} H(3 - x)$

Answer: 1

e. [2 points] $\lim_{h \to 0} \dfrac{H(2.5 + h) - H(2.5)}{h}$

Answer: 2

Define the function $J(x) = 3H\big(-2x\big) - 1$. You do not need to show work in the next two parts.

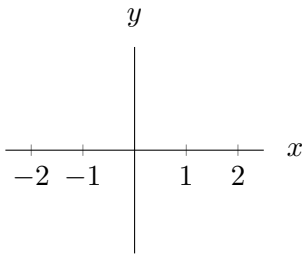
f. [2 points] Where does the function $J(x)$ have a vertical asymptote? At $x =$ $-1/2$.

g. [2 points] Where does the function $J(x)$ have a horizontal asymptote? At $y =$ 5.

6. [7 points] For each part below, sketch the graph of a function that satisfies the given properties, or, if there is no function satisfying all the properties in that part, write DNE instead.

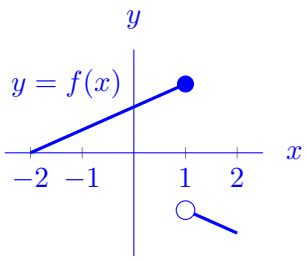
Any graphs you draw should have axes like those shown to the right. Make sure your graphs are clear and unambiguous, with any important values marked on the axes.

Note: If DNE is written, then any graph you have drawn in that part will not be graded.



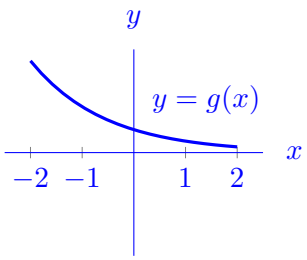
a. [2 points] A function $f(x)$ that satisfies $\lim_{x\to 1^-} f(x) = f(1)$ but that is not continuous at $x = 1$

Solution:



b. [2 points] A function $g(x)$ that is positive on $-2 < x < 2$ and such that $g'(x)$ is negative on $-2 < x < 2$

Solution:



c. [3 points] A continuous function $h(x)$ that is also invertible and that has an average rate of change of zero on the interval $[-1, 1]$

Solution: DNE.

(Note that an average rate of change of zero means $\dfrac{h(1) - h(-1)}{1 - (-1)} = 0$. But if we simplify, this means $h(1) = h(-1)$, so $h(x)$ can't be invertible.)

7. [10 points] Kimoi is going to hold an Autumn Festival at her store next year. She wants to make and provide free carrot juice to her customers using carrots from her garden.

- Let $c(w)$ be the amount of carrots, measured in pounds, that grow when she gives her carrot garden w gallons of water during the growing season.
- Let $j(v)$ be the amount of carrot juice, measured in gallons, that she can make from v pounds of carrots.

The functions $c(w)$ and $j(v)$ are both invertible and differentiable.

a. [2 points] Write a complete sentence that gives a practical interpretation of the equation

$$c^{-1}(38) = 620.$$

Solution: To grow 38 pounds of carrots, Kimoi needs to give her garden 620 gallons of water.

b. [2 points] Write an equation involving c, j, and/or their inverses that represents the following statement.

If Kimoi gives her garden 1120 gallons of water, then she can produce 10 gallons of carrot juice.

Answer: $j(c(1120)) = 10$

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$c'(900) = 0.2.$$

If Kimoi gives her carrot garden 903 gallons of water rather than 900 gallons of water, then...

Solution: ... she will be able to grow about 0.6 additional pounds of carrots.

d. [3 points] Write the roman numeral of the <u>one</u> sentence below that gives a valid interpretation of the equation

$$(j^{-1})'(10) = 18.$$

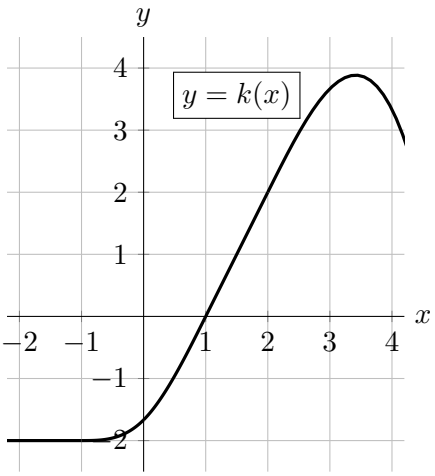
i. If Kimoi has 11 pounds of carrots instead of 10 pounds, then she can make approximately 18 more gallons of carrot juice.

ii. To make 10 gallons of carrot juice instead of 9.5, Kimoi will need to give her garden about 9 additional gallons of water.

iii. If Kimoi increases her carrot yield from 18 pounds to 19 pounds, then she can make about 10 more gallons of carrot juice.

iv. If Kimoi wants to increase the amount of carrot juice she makes from 10 gallons to 10.5 gallons, then she needs about 9 more pounds of carrots.

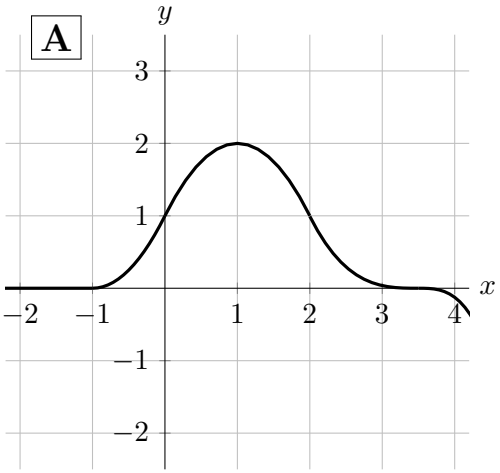
8. [4 points] A portion of the graph of the differentiable function $k(x)$ is shown below. Note that $k(x)$ is linear on $-2 \le x \le -1$ and $1 \le x \le 2$.



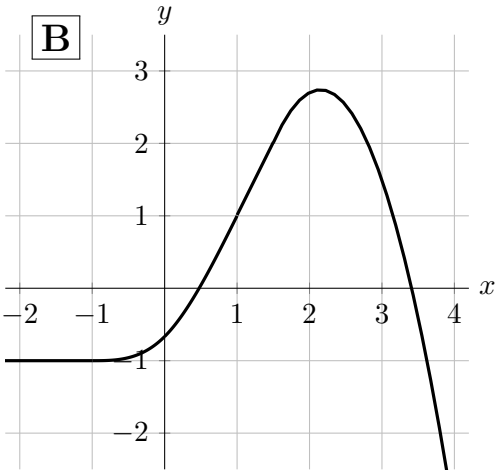
Which of the following graphs could be the graph of $k'(x)$, the derivative of $k(x)$? There is exactly one correct answer. Write the letter of your answer.

Note: If you are having trouble viewing the graphs, zooming in on the pdf file may help.

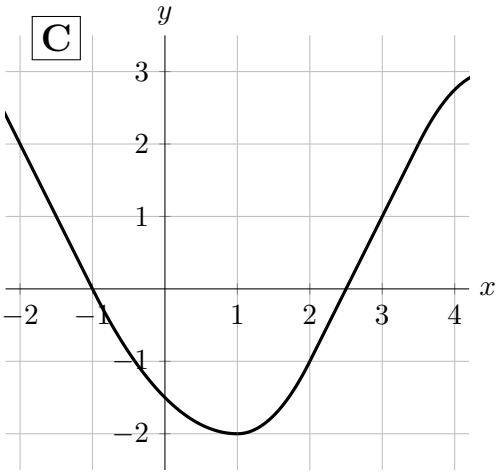
A



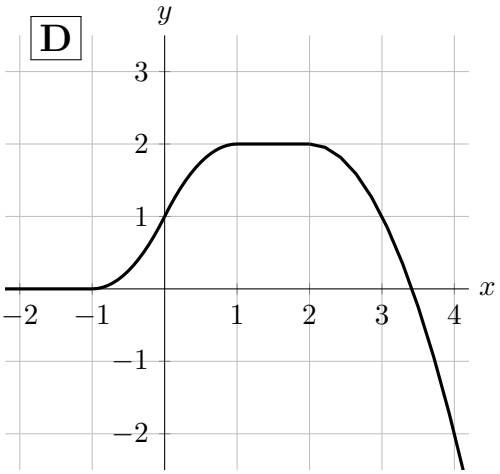
B



C



D



Answer: D