Math 115 — Second Midterm — November 8, 2021

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 9 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

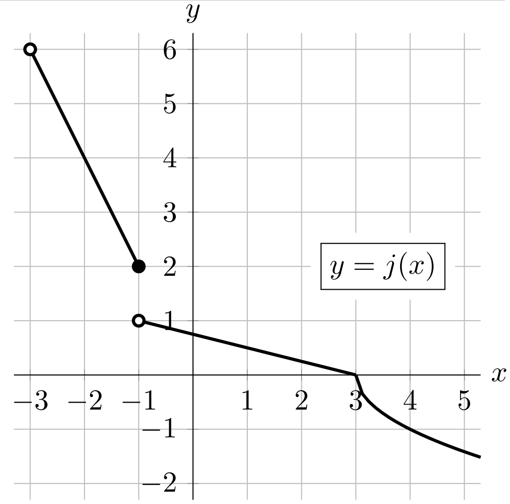
Problem	Points	Score
1	10	
2	11	
3	15	
4	12	
5	8	

Problem	Points	Score
6	8	
7	6	
8	8	
9	12	
Total	90	

1. [10 points]

A portion of the graph of the function $j(x)$, whose domain is $(-3, \infty)$, is shown to the right. Note that:

- $j(x)$ is linear on $(-3, -1]$ and on $(-1, 3]$.
- On the interval $[3, \infty)$, the function $j(x)$ is given by the formula $-\sqrt{x-3}$.



For parts **a.–c.**, find the **exact** values, or write NEI if there is not enough information to do so, or write DNE if the value does not exist. Your answers should not include the letter j but you do not need to simplify. Show work.

a. [2 points] Find $j'(4)$.

Solution: On $[3, \infty)$, $j(x) = -(x-3)^{1/2}$ so $j'(x) = \frac{-1}{2}(x-3)^{-1/2}$ and Using the chain rule,

$$\left[-\sqrt{x-3}\right]' = -\frac{1}{2}(x-3)^{-1/2}$$

so

$$j'(4) = -\frac{1}{2}(4-3)^{-1/2} = -\frac{1}{2}$$

Answer: $j'(4) =$ $-1/2$

b. [3 points] Let $A(x) = \dfrac{x}{j(x)}$. Find $A'(1)$.

Solution: Using the quotient rule,

$$A'(x) = \frac{j(x) - xj'(x)}{(j(x))^2}$$

$$A'(1) = \frac{j(1) - j'(1)}{(j(1))^2} = \frac{\frac{1}{2} - \left(-\frac{1}{4}\right)}{\left(\frac{1}{2}\right)^2} = 3$$

Answer: $A'(1) =$ 3

c. [3 points] Let $B(x) = 2^{j(x)}$. Find $B'(-2)$.

Solution: Using the chain rule,

$$B'(x) = \ln(2)2^{j(x)}j'(x)$$

$$B'(-2) = \ln(2)2^{j(-2)}j'(-2)$$

$$= \ln(2)2^4(-2) = -32\ln(2).$$

Answer: $B'(-2) =$ $-32\ln(2)$

d. [2 points] On which of the following intervals does $j(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers. You do not need to show work for this part.

$[-1, 2]$ $[0, 5]$ $\boxed{[3, 5]}$ NONE OF THESE

2. [11 points]

a. [6 points] Let $f(x)$ be a continuous function defined for all real numbers and suppose that $f'(x)$, the derivative of $f(x)$, is given by

$$f'(x) = \frac{(x-2)(x+3)}{|x|}.$$

Find the exact x-coordinates of all local minima and local maxima of $f(x)$. If there are none of a particular type, write NONE. You must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

Solution: The critical points of f are where $f'(x) = 0$, which occurs at $x = 2$ and $x = -3$, and where f' DNE, which occurs at $x = 0$.

(checking signs for 1st Derivative Test)	$x < -3$	$-3 < x < 0$	$0 < x < 2$	$2 < x$
$(x-2)$	$-$	$-$	$-$	$+$
$(x+3)$	$-$	$+$	$+$	$+$
$\|x\|$	$+$	$+$	$+$	$+$
$f'(x) = \frac{(x-2)(x+3)}{\|x\|}$	$\frac{-\cdot-}{+} = +$	$\frac{-\cdot+}{+} = -$	$\frac{-\cdot+}{+} = -$	$\frac{+\cdot+}{+} = +$

This gives the following number line for $f'(x)$:

$f'(x)$	$\frac{-\cdot-}{+} = +$	-3	$\frac{-\cdot+}{+} = -$	0	$\frac{-\cdot+}{+} = -$	2	$\frac{+\cdot+}{+} = +$

By the First Derivative Test, $f(x)$ has a local max at $x = -3$ and a local min at $x = 2$. There is no local extremum at $x = 0$.

Note: Though it doesn't change the points you need to consider on your number line, there is actually no function f that satisfies these conditions! To find out why, take Math 116.

Answer: Local max(es) at $x =$ -3 and Local min(s) at $x =$ 2

b. [5 points] Let $g(x)$ be a different continuous function defined for all real numbers and suppose that $g''(x)$, the second derivative of $g(x)$, is given by

$$g''(x) = 2^x(x-1)^2(x+5).$$

Find the exact x-coordinates of all inflection points of $g(x)$, or write NONE if there are none. You must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

Solution: We start by finding any values of x for which $g''(x) = 0$ or g'' DNE, and find $x = 1$ and $x = -5$. Now we check to see whether the concavity of g changes at these points:

$g''(x)$	$+\cdot+\cdot- = -$	-5	$+\cdot+\cdot+ = +$	1	$+\cdot+\cdot+ = +$

Answer: Inflection point(s) at $x =$ -5

3. [15 points] Let the differentiable function $h(t)$ represent the height in inches (in) of a toy airplane above the ground at time t seconds (sec). Below is a table of some values for $h(t)$ and $h'(t)$. Assume that $h(t)$ is invertible, and that $h'(t)$ is differentiable for $t > 0$.

t	0	2	4	6	8
$h(t)$	28	19	11	8	4
$h'(t)$	-5	-4	-2	-1.5	-0.5

For parts **a.**–**d.**, you do not need to show work, but partial credit can be earned from work shown. You do not need to simplify numerical answers.

a. [3 points] Approximate $h''(8)$. Include units.

Solution:

$$h''(8) \approx \frac{h'(8) - h'(6)}{8-6} = \frac{-0.5 - (-1.5)}{2} = \frac{1}{2}.$$

Answer: 0.5 **in/sec^2**

b. [3 points] Find a formula for the linear approximation $L(t)$ to the function $h(t)$ at $t = 2$.

Solution: $L(t) = h(2) + h'(2)(t-2) = 19 - 4(t-2) = -4t + 27$

Answer: $L(t) =$ $19 - 4(t-2) = -4t + 27$

c. [2 points] Use your answer from the previous part to approximate $h(1.9)$. Include units.

Solution: $h(1.9) \approx L(1.9) = 19 - 4(1.9 - 2) = 19 - 4(-0.1) = 19.4$

Answer: 19.4 **inches**

d. [2 points] Compute the **exact** value of $(h^{-1})'(8)$. (You do not need to include units.)

Solution: $(h^{-1})'(8) = \dfrac{1}{h'(h^{-1}(8))} = \dfrac{1}{h'(6)} = \dfrac{1}{-1.5}$.

Answer: $\dfrac{1}{-1.5} = -\dfrac{2}{3}$

e. [3 points] Suppose that $(h^{-1})'(3) = -9$. Complete the following sentence to give a practical interpretation of this equation.

When the toy airplane is at a height of 3 inches, to descend an additional 0.1 inches . . .

Solution: ... will take about 0.9 seconds.

f. [2 points] Note that $h(t)$ satisfies the hypotheses of the Mean Value Theorem on $[0, 8]$. Complete the following sentence about what the conclusion of this theorem implies is true.

At some time between $t = 0$ and $t = 8$, the height of the toy airplane is . . .

(circle one) INCREASING **[DECREASING]** at a rate of **3** in/sec.

4. [12 points] Consider the continuous, piecewise-defined function $r(x)$ given as follows:

$$r(x) = \begin{cases} -x^2 - 2x & \text{if } x < 0 \\ xe^{-x} & \text{if } x \geq 0 \end{cases}.$$

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

It may be helpful to note that $e \approx 2.72$ and/or that $\frac{1}{e} \approx 0.37$.

a. [4 points] Find all critical points of $r(x)$. Show all your work.

Solution:

$$r'(x) = \begin{cases} -2x - 2 & \text{if } x < 0 \\ -xe^{-x} + e^{-x} = e^{-x}(1-x) & \text{if } x > 0 \end{cases}$$

To find critical points, we need to find where $r'(x) = 0$ and where $r(x)$ is not differentiable. Using the first piece of $r'(x)$, we have $-2x - 2 = 0$ when $x = -1$. (This is less than 0 so is indeed in the domain of the first piece of the function.) From the second piece, we have $e^{-x}(x-1) = 0$ when $x = 1$. (This is greater than 0 so is indeed in the domain of the second piece of the function.)

Finally, we need to check differentiability at $x = 0$. We are told r is continuous (though we can also quickly check that both pieces of $r(x)$ evaluate to 0 when $x = 0$). However, note that plugging $x = 0$ into the two pieces of $r'(x)$ gives -2 for the first piece and e^{-1} for the second piece. Hence there is a sharp corner at $x = 0$, which is therefore also a critical point of $r(x)$. so $r'(x) = 0$ at $x = -1$ and 1. Also, $r'(x)$ DNE at 0 because $-2 \neq 1$.

Answer: Critical point(s) at $x =$ $-1, 0, 1$

b. [4 points] Find the x-coordinates of all global minimum(s) and global maximum(s) of $r(x)$ **on the interval** $[-2, 1]$. If there are none of a particular type, write NONE.

Solution: Since $r(x)$ is continuous on the closed interval $[-2, 1]$, the Extreme Value Theorem guarantees a global max and a global min. To find these, we create a table of values of $r(x)$ for all critical points in and endpoints of the domain $[-2, 1]$.

We see that the maximum value of 1 is attained at $x = -1$ and the minimum value of 0 is attained at both $x = -2$ and $x = 0$.

x	$r(x)$
-2	0
-1	1
0	0
1	$e^{-1} = \frac{1}{e} \approx 0.37$

Answer: Global max(es) at $x =$ -1

Answer: Global min(s) at $x =$ -2 **and** 0

c. [4 points] Find the x-coordinates of all global minimum(s) and global maximum(s) of $r(x)$ **on the interval** $(-\infty, \infty)$. If there are none of a particular type, write NONE.

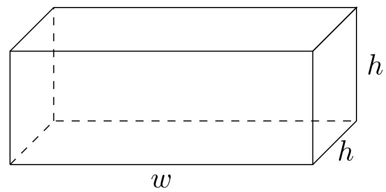
Solution: We create a table showing the value of $r(x)$ at all critical points along with the end behavior of $r(x)$ on the domain $(-\infty, \infty)$. Since $r(x)$ decreases without bound as $x \to -\infty$, there is no global minimum of $r(x)$. However, $r(x)$ attains a global maximum value of 1 at $x = -1$.

x	$r(x)$
$\lim_{x\to-\infty}$	$-\infty$ (DNE)
-1	1
0	0
1	$e^{-1} = \frac{1}{e} \approx 0.37$
$\lim_{x\to\infty}$	0

Answer: Global max(es) at $x =$ -1

Answer: Global min(s) at $x =$ NONE

5. [8 points] Aziz is starting a donut shop. He wants to build a display case to show off his donuts. He wants a case of width w feet, height h feet, and length h feet, so that, as shown in the diagram below, the left and right sides are squares. The top and front of the case will be made of glass, while the square sides, back, and bottom will be made of metal. Glass costs 2 dollars per square foot, and metal costs 4 dollars per square foot. Aziz plans to spend exactly 80 dollars on the display case.



a. [3 points] Write a formula for w in terms of h.

Solution:

$$80 = 2(2hw) + 2(4h^2) + 2(4hw)$$
$$80 - 8h^2 = w(4h + 8h)$$
$$w = \frac{80 - 8h^2}{12h} = \frac{20 - 2h^2}{3h} = \frac{20}{3h} - \frac{2h}{3}$$

b. [2 points] Aziz wants to maximize the volume of the display case. Find a formula for the function $V(h)$ which gives the volume in cubic feet of the display case in terms of h only.
Your formula should not include the letter w.

Solution:

$$V = h^2 w$$
$$V(h) = h^2\left(\frac{80 - 8h^2}{12h}\right) = \frac{80h - 8h^3}{12}$$

c. [3 points] What is the domain of the function $V(h)$ in the context of this problem?

Solution:

$$0 \leq w = \frac{80 - 8h^2}{12h}$$
$$0 \leq 80 - 8h^2$$
$$8h^2 \leq 80$$
$$h \leq \sqrt{\frac{80}{8}}$$

Note that h cannot be 0. So domain is $0 < h \leq \sqrt{10}$ or $(0, \sqrt{10}]$. It's also reasonable to use a domain of $0 < h < \sqrt{10}$ instead.

6. [8 points] Aziz is trying to finalize his donut frosting recipe, which uses only butter and sugar. He makes frosting in 3-pound batches, so if he uses x pounds of butter and y pounds of sugar, he needs

$$x + y = 3.$$

He believes that the number of donuts D he will sell each year, in thousands, will depend on his frosting recipe, and in particular, that D can be modeled by the function

$$D = \frac{x^3}{3} + 2xy + 3y - 8.$$

What values of x and y will maximize the number of donuts he sells? Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the number of donuts sold.

Solution: The constraint is $3 = x + y$, so $y = 3 - x$. Plugging in gives

$$D(x) = \frac{x^3}{3} + 2x(3-x) + 3(3-x) - 8 = \frac{x^3}{3} - 2x^2 + 3x + 1.$$

Then taking the derivative and setting it equal to zero, we have

$$D'(x) = x^2 - 4x + 3 = (x-3)(x-1)$$
$$0 = (x-3)(x-1).$$

D' always exists, so the only critical points are $x = 1, 3$, and domain is $[0, 3]$, since we need $x \geq 0$ and $y \geq 0$. Plugging in,

x	$D(x)$
0	1
1	$\frac{1}{3} + 2$
3	1

Thus the maximum occurs when $x = 1$ and $y = 2$.

Alternate Solution: The constraint is $3 = x + y$, so $x = 3 - y$. Plugging in gives

$$D(y) = \frac{(3-y)^3}{3} + 2(3-y)y + 3y - 8.$$

Then

$$D'(y) = -(3-y)^2 + 2(3-y) - 2y + 3$$
$$0 = -(y^2 - 6y + 9) - 4y + 9$$
$$0 = -y^2 + 2y$$
$$0 = -y(y-2)$$

So critical points are $y = 0, 2$, and domain is $[0, 3]$ since we need $y \geq 0$ and $x \geq 0$. Plugging in,

x	$D(x)$
0	1
2	$\frac{1}{3} + 2$
3	1

Thus the maximum occurs when $x = 1$ and $y = 2$.

7. [6 points] Define the piecewise function $g(x)$ as below, where a and b are constants.

$$g(x) = \begin{cases} a + b\sin(\pi(x+2)) & x \le -2 \\ -3(x+2)+4 & x > -2 \end{cases}$$

Find one pair of **exact** values for a and b such that $g(x)$ is differentiable, or write NONE if there are none. You do not need to simplify your answers but be sure your work is clear.

Solution: First, we need $g(x)$ to be continuous at $x = -2$.

$$g(-2) = \lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} a + b\sin(\pi(x+2)) = a + b\sin(\pi(-2+2)) = a + b\sin(0) = a.$$

$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} -3(x+2)+4 = -3(-2+2)+4 = 4.$$

So in order for $g(x)$ to be continuous at $x = -2$, we must have $\boxed{a = 4}$ (and in that case, $g(x)$ is indeed continuous at $x = -2$).

For differentiability, we also need the slope on each side of the point at $x = -2$ to match up so that there is not a sharp corner. We have

$$g'(x) = \begin{cases} \pi \cdot b\cos(\pi(x+2)) & x < -2 \\ -3 & x > -2 \end{cases}$$

Plugging in $x = -2$ to the first piece gives $\pi \cdot b\cos(\pi(-2+2)) = \pi \cdot b\cos(0) = \pi \cdot b$. So differentiability requires $\pi \cdot b = -3$ and therefore $\boxed{b = -3/\pi}$.

Answer: $a =$ 4 and $b =$ $-3/\pi$

8. [8 points]

a. [5 points] Consider the curve $\mathcal{C}$ defined by the equation $\ln(x^2) + y = e^{4y}$.

For this curve $\mathcal{C}$, find a formula for $\frac{dy}{dx}$ in terms of x and y. Clearly show every step of your work.

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(\ln(x^2)+y\right) &= \frac{d}{dx}\left(e^{4y}\right)\\
\frac{2x}{x^2} + \frac{dy}{dx} &= 4e^{4y}\frac{dy}{dx}\\
\frac{2}{x} &= 4e^{4y}\frac{dy}{dx} - \frac{dy}{dx}\\
\frac{2}{x} &= \frac{dy}{dx}\left(4e^{4y}-1\right)\\
\frac{2}{x\left(4e^{4y}-1\right)} &= \frac{dy}{dx}
\end{aligned}$$

b. [3 points] Let $\mathcal{D}$ be a different implicitly defined curve. The curve $\mathcal{D}$ passes through the point $(2, 1)$ and satisfies

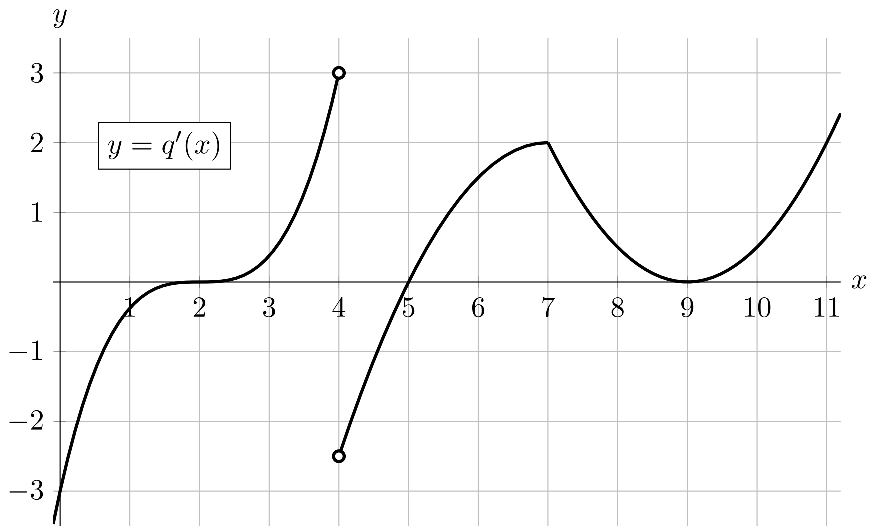
$$\frac{dy}{dx} = \frac{-2x - y}{x + 3y^2 - 1}.$$

Write an *equation* for the tangent line to the curve $\mathcal{D}$ at the point $(2, 1)$. Show your work.

Solution: We are given the coordinates of a point on the line so need only to find the slope, which is given by $\frac{dy}{dx}$. Plugging in the point $(2, 1)$ we find that the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2(2)-1}{2+3(1)^2-1} = \frac{-5}{4}$ so $y = 1 - \frac{5}{4}(x-2)$. Using point-slope form we find that an equation for the tangent line at $(2, 1)$ is

$$y = 1 - \frac{5}{4}(x-2).$$

9. [12 points] Let $q(x)$ be a continuous function which is defined for all real numbers. A portion of the graph of $q'(x)$, **the derivative of $q(x)$**, is shown below.



For each of the following, circle all correct choices.

a. [2 points] On which of the following interval(s) is $q(x)$ increasing?

$(0,2)$ $\boxed{(2,4)}$ $\boxed{(7,9)}$ NONE OF THESE

b. [2 points] Which of the following are critical point(s) of $q(x)$?

$\boxed{x=4}$ $\boxed{x=5}$ $x=7$ NONE OF THESE

c. [2 points] At which of the following value(s) of x does $q(x)$ have a local maximum?

$\boxed{x=4}$ $x=5$ $x=7$ NONE OF THESE

d. [2 points] On which of the following interval(s) is $q''(x)$ positive?

$\boxed{(0,2)}$ $\boxed{(2,4)}$ $(7,9)$ NONE OF THESE

e. [2 points] At which of the following value(s) of x does $q(x)$ have an inflection point?

$x=2$ $\boxed{x=7}$ $\boxed{x=9}$ NONE OF THESE

f. [2 points] At which of the following value(s) of x does $q'(x)$ have an inflection point?

$\boxed{x=2}$ $\boxed{x=7}$ $x=9$ NONE OF THESE