Math 115 — First Midterm — October 3, 2022

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 8 pages including this cover. There are 9 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. Scratchwork on pages other than those in this exam will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	7	
2	6	
3	5	
4	7	
5	8	

Problem	Points	Score
6	11	
7	3	
8	5	
9	8	
Total	60	

1. [7 points] Let

$$g(w) = 1 - \frac{e^w}{6w}.$$

a. [2 points] Evaluate each of the limits below. If a limit does not exist, including if it diverges to $\pm\infty$, write DNE. You do not need to show work.

i. $\lim\limits_{w\to\infty} g(w)$

Answer: $-\infty$ **(DNE)**

ii. $\lim\limits_{w\to-\infty} g(w)$

Answer: 1

b. [5 points] Use the limit definition of the derivative to write an explicit expression for $g'(3)$. *Your answer should not involve the letter g. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $g'(3) =$ $\boxed{\lim\limits_{h\to 0} \frac{\left(1 - \frac{e^{3+h}}{6(3+h)}\right) - \left(1 - \frac{e^3}{6\cdot 3}\right)}{h}}$

2. [6 points] You are an intern at A&B, Alice & Bob Inc. It's the year 2055, and you're headed to Mars to help the newest A&B store, which will open there on January 1^{st}, 2056.

a. [3 points] The daily sales of space suits, in thousands, at the new store d Earth days after it opens can be modeled by the sinusoidal function

$$S(d) = 16\sin\left(\frac{2\pi}{687}\cdot d\right) + 17.$$

i. The function $S(d)$ has a period of one Mars year. Use this information to find the length of a Mars year in units of Earth days.

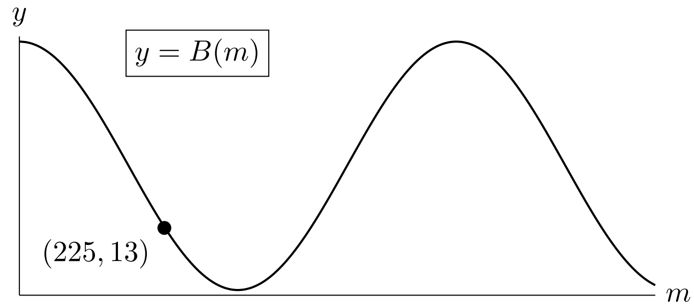
Answer: One Mars year is **687** Earth days.

ii. According to this model, what are the minimum and maximum daily sales, in thousands, of A&B space suits on Mars?

minimum sales of **1** thousand suits

maximum sales of **33** thousand suits

b. [3 points] The daily sales of space boots, in thousands, at the new store m Mars days after it opens can be modeled by a different sinusoidal function $B(m)$, which also has a period of one Mars year, which is 670 Mars days. The graph of $B(m)$ is given below. Note that a maximum occurs at $m = 0$.



The first time that daily sales of space boots equals 13,000 is $m = 225$ Mars days after the store opens, as shown on the graph. Find the next two values of m at which daily sales of space boots will equal 13,000 according to this model. You do not need to simplify your answers.

Solution: We know that the period is equal to 670 and $m = 0$ is a maximum. Therefore, using the symmetry of the graph, the next time sales will be 13,000 is

$$670 - 225 = 445.$$

To get the third time when sales are 13,000 we should just add a period to the first solution:

$$225 + 670 = 895.$$

Answer: **445** and **895**

3. [5 points] The function $y(t)$, given to the right, gives the mass, in milligrams, of a yeast colony t hours after an experiment begins, where A, B, and C are constants.

$$y(t) = \begin{cases} \dfrac{A}{1+e^{-Ct}} & 0 \le t < 4 \\ 12B^t & t \ge 4 \end{cases}$$

Find the values of A, B, and C such that all of the following hold:

- $\lim\limits_{t\to 0^+} y(t) = 8$,
- the yeast colony's mass decays by 2% each hour after $t = 4$, and
- $y(t)$ is continuous at $t = 4$.

Show your work, and give your answers in **exact form**.

Solution: Because $\frac{A}{1+e^{-Ct}}$ is a continuous function, the first point gives

$$8 = \lim_{t\to 0^+} \frac{A}{1+e^{-Ct}} = \frac{A}{1+e^{-C\cdot 0}} = \frac{A}{1+1},$$

so $A = 16$. The second point gives $B = 0.98$.
Finally, the third point means we must have $\lim\limits_{t\to 4^-} y(t) = \lim\limits_{t\to 4^+} y(t) = y(4)$. So we must have

$$\frac{16}{1+e^{-C\cdot 4}} = 12\cdot(0.98)^4 \quad \text{so} \quad \frac{16}{12\cdot(0.98)^4} - 1 = e^{-C\cdot 4}.$$

Taking ln of both sides and then dividing by -4, we get $C = -\frac{1}{4}\ln\left(\dfrac{16}{12\cdot(0.98)^4} - 1\right)$.

Answers: $A =$ 16 $B =$ 0.98 $C = -\dfrac{1}{4}\ln\left(\frac{16}{12\cdot(0.98)^4} - 1\right)$

4. [7 points] Consider the rational function $r(x) = \dfrac{x(x-1)(x+4)^2}{(x^2-1)(x+4)}$.

a. [5 points]

i. Find the equations of any horizontal asymptotes of $r(x)$ or write NONE if there are none.

Answer: **NONE**

ii. Find all the zeros of $r(x)$, or write NONE if there are none.

Answer: 0

iii. Find all numbers c such that the limit $\lim\limits_{x\to c} r(x)$ exists but $r(c)$ is *not* defined.

Answer: $-4, 1$

b. [2 points] Find a linear function $h(x)$ such that the function $r(x)\cdot h(x)$ has no vertical asymptotes.

Answer: $h(x) =$ $x+1$

5. [8 points] The amount of power produced by a wind turbine depends on the speed of the wind. In particular, suppose $P(s)$ is the power, in megajoules per hour (MJ/h), produced by the turbine when the speed of the wind is s kilometers per hour (km/h). Also suppose that $W(t)$ gives the wind speed, in km/h, at the turbine's location t hours after noon on a certain day.

Assume that $P(s)$ is invertible, and that both $P(s)$ and $W(t)$ are differentiable.

a. [2 points] Use a complete sentence to give a practical interpretation of the equation

$$P(W(0)) = 8.$$

Solution: At noon, the turbine produces 8 MJ/h of power.

b. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$W'(4) = 18.$$

From 4:00*pm to* 4:10*pm, ...*

Solution: the wind speed at the turbine's location increases by approximately 3 km/h.

c. [3 points] Circle the one statement below that is best supported by the equation

$$\left(P^{-1}\right)'(13) = 2.9.$$

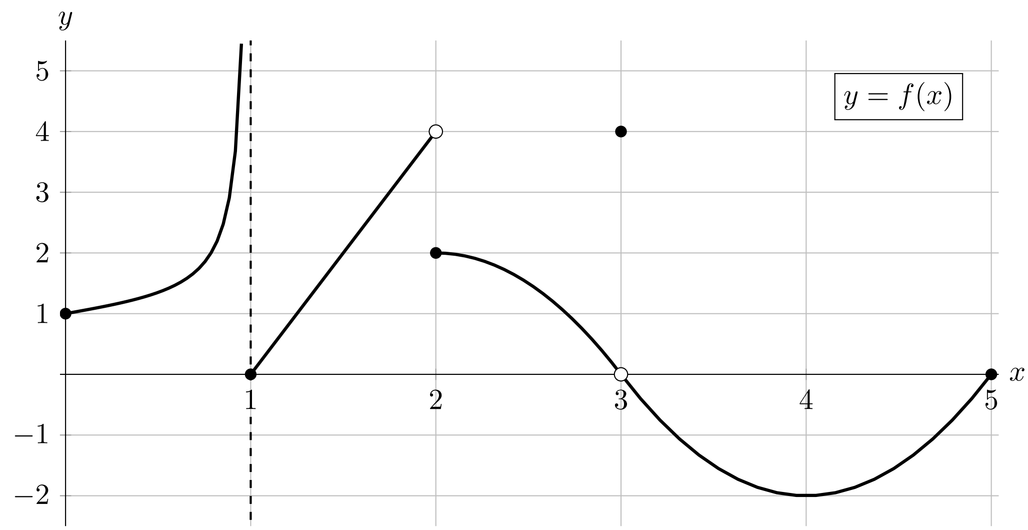
i. When the turbine is generating 13 MJ/h of power, an increase of one km/h in wind speed will produce approximately 2.9 MJ/h more power.

ii. *If the turbine is producing 13 MJ/h of power, the wind speed must increase by approximately 2.9 km/h to produce an additional MJ/h of power.*

iii. If the wind is blowing at 13 km/h and increases to 14 km/h, the power produced by the turbine will increase by about 2.9 MJ/h.

iv. If the wind speed is 13 km/h, the power generation of the turbine will increase by one MJ/h if the wind speed increases to about 15.9 km/h.

6. [11 points] Below is a portion of the graph of an even function $f(x)$, which has domain $(-\infty,\infty)$ even though the graph below only shows the function on the interval $[0,5]$. Note that $f(x)$ has a vertical asymptote at $x=1$.



a. [1 point] At which of the following values of x is $f(x)$ continuous? Circle all correct answers.

$x=1$ $\qquad$ $x=2$ $\qquad$ $x=3$ $\qquad$ $\boxed{x=4}$ $\qquad$ NONE OF THESE

b. [8 points] Find the **exact** numerical value of each expression below, if possible. For any values that do not exist, including if they are limits that diverge to $\pm\infty$, write DNE. If there is not enough information to find a given value or determine whether it exists, write NEI.
You do not need to show work. As a reminder, $f(x)$ is an even function.

$f(f(3)) = -2$

$\displaystyle\lim_{x\to 0^-} f(x) = 1$

$\displaystyle\lim_{x\to 2} f(x) =$ DNE

$\displaystyle\lim_{x\to 6^+} \frac{f(x-2)}{f\left(\frac{x}{3}\right)} = -1$

$\displaystyle\lim_{x\to 3} f(x) = 0$

$\displaystyle\lim_{x\to 2^-} f(-x) = 4$

$\displaystyle\lim_{x\to 1^-} \frac{1}{f(x)} = 0$

$\displaystyle\lim_{h\to 0} \frac{f(1.5+h)-f(1.5)}{h} = 4$

c. [2 points] Consider the function $G(x) = -f(x+3)$. Which of the following must be a vertical asymptote of $G(x)$? There is only one correct answer.

$x=-3$ $\qquad$ $\boxed{x=-2}$ $\qquad$ $x=-1$ $\qquad$ $x=1$ $\qquad$ $x=4$

7. [3 points] Assume that $k(t)$ is a differentiable function defined for all t, and that the tangent line to the graph of $k(t)$ at $t = 2$ passes through the points $(1, 10)$ and $(4, 19)$. Find the values of $k(2)$ and $k'(2)$. You do not need to show work, but limited partial credit may be earned for work shown.

Solution: The slope of the tangent line is equal to $\frac{19-10}{4-1} = \frac{9}{3} = 3$, so this is $k'(2)$. Then the tangent line must pass through $(2, 13)$, which means we must have $k(2) = 13$.

Answer: $k(2) =$ **13** and $k'(2) =$ **3**

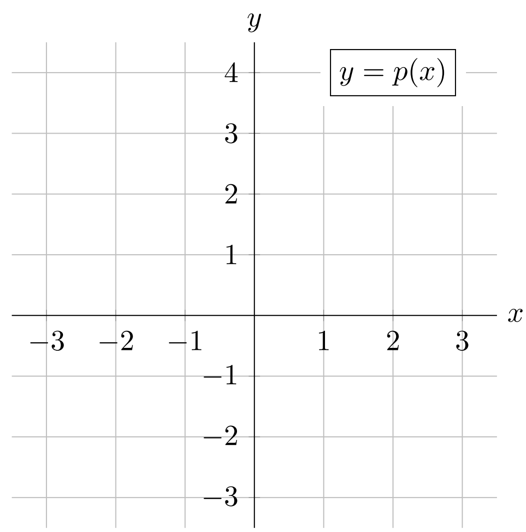
8. [5 points] For each part below, carefully draw the graph of a single function on the given axes that satisfies the given conditions, or, if no such function exists, write DNE.

a. [2 points]

A function $p(x)$ such that

- $p(x)$ is defined for all $-3 < x < 3$,
- $\dfrac{p(2) - p(-2)}{2 - (-2)} = 0$, and
- $p(x)$ is invertible.

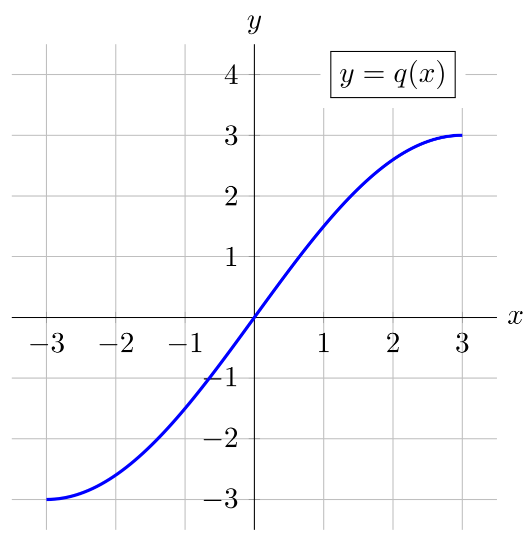
Solution: DNE, because we must have $p(2) = p(-2)$ and so $p(x)$ does not pass the horizontal line test.



b. [3 points]

A function $q(x)$ such that

- $q(x)$ is defined for all $-3 < x < 3$,
- $q(x)$ is increasing on $(-3, 3)$,
- $q(x)$ is concave up on $(-3, 0)$, and
- $q(x)$ is an odd function.

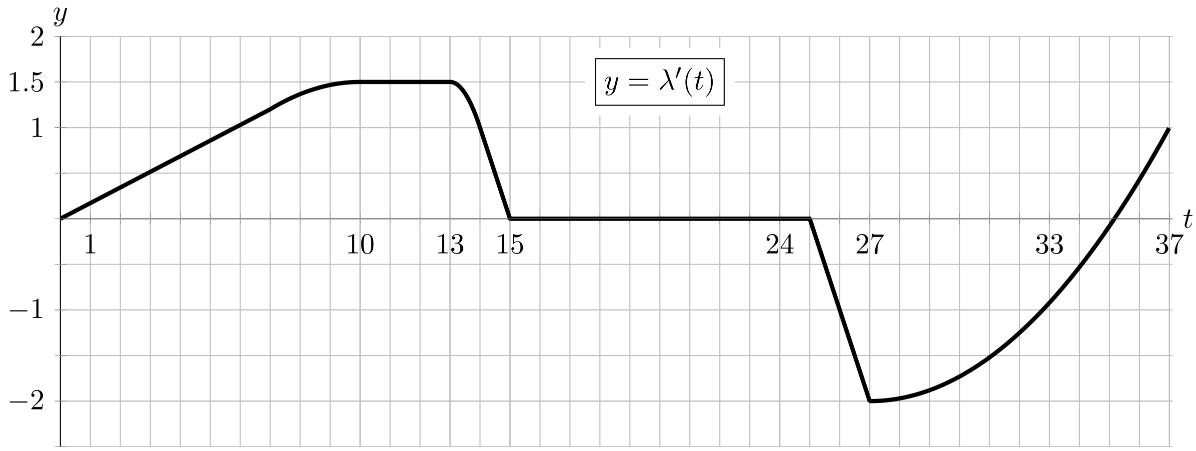


9. [8 points] The *Lambda* app, developed by your friends, displays information about a train that departed the Detroit station at noon and is traveling on the track between Detroit and Ann Arbor.

The app shows you several values of $\lambda(t)$, the differentiable function that gives the distance along the track, in kilometers, from the Detroit station to the train t minutes after noon:

t	37	39	41	43	45
$\lambda(t)$	16	16	24	30	36

The graph of $\lambda'(t)$, the <u>derivative</u> of $\lambda(t)$, for $0 \leqslant t \leqslant 37$, is also shown:



For parts **a.** and **b.**, you may estimate values from the graph as needed.

a. [1 point] Find all times t for $0 < t < 37$ when the train is traveling at its maximum *velocity*. Give your answer as value(s) and/or interval(s) of t.

Answer: $(10, 13)$

b. [1 point] Find all times t for $0 < t < 37$ when the train is traveling at its maximum *speed*. Give your answer as value(s) and/or interval(s) of t.

Answer: 27

c. [2 points] Find the average velocity of the train between 12:00pm and 12:45pm. Include units.

Answer: $\frac{36}{45} = \frac{4}{5}$ **km per minute**

d. [2 points] Estimate the instantaneous velocity of the train at $t = 41$. Include units.

Solution: $\frac{24-16}{41-39} = \frac{8}{2} = 4$, or $\frac{30-24}{43-41} = \frac{6}{2} = 3$, or the average of these, or $\frac{30-16}{43-39} = \frac{14}{4}$ km per minute

Answer: 3.5 **km per minute**

e. [2 points] During which of the following time intervals is the train stopped for the entire time? Circle <u>all</u> correct choices.

$(3, 5)$ $(10, 13)$ $\boxed{(18, 21)}$ $(41, 45)$ NONE OF THESE