Math 115 — Second Midterm — November 7, 2022

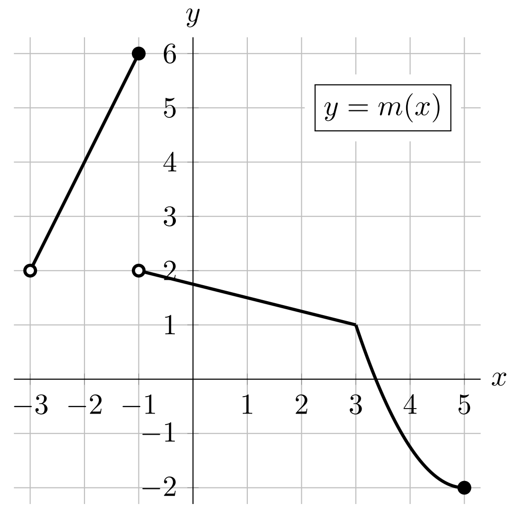
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	9	
3	8	
4	4	
5	7	
6	6	

Problem	Points	Score
7	8	
8	7	
9	7	
10	7	
11	8	
Total	80	

1. [9 points]



The graph of the function $m(x)$ is shown to the right. Note that:

- $m(x)$ is linear on $(-3,-1]$ and on $(-1,3]$,
- $m(x)$ is quadratic on $[3,5]$, and
- there is a corner at $x=3$.

For parts **a.–d.**, find the **exact** values, or write DNE if the value does not exist. Your answers should not include the letter m but you do not need to simplify.

a. [1 point] Find $m''(1)$.

Answer: $m''(1) =$ 0

b. [2 points] Let $A(x) = \dfrac{m(x)}{x}$. Find $A'(-2)$.

Solution: $A'(x) = \dfrac{xm'(x)-m(x)}{x^2}$ so $A'(-2) = \dfrac{-2(2)-4}{(-2)^2} = \dfrac{-8}{4} = -2$

Answer: $A'(-2) =$ -2

c. [2 points] Let $B(x) = m(x)\ln(3x)$. Find $B'(1)$.

Solution: $B'(x) = m(x)\dfrac{1}{3x}\cdot 3 + m'(x)\ln(3x)$ so $B'(1) = 1.5\cdot\frac{1}{1} - \dfrac{1}{4}\ln(3)$

Answer: $B'(1) =$ $\dfrac{3}{2} - \dfrac{1}{4}\ln(3)$

d. [2 points] Let $C(x) = m^{-1}(x)$. Find $C'(1)$.

Solution: $C'(x) = \dfrac{1}{m'(m^{-1}(x))}$, so $C'(1)$ would be $\dfrac{1}{m'(m^{-1}(1))} = \dfrac{1}{m'(3)}$, but $m'(3)$ doesn't exist. Indeed, the graph of $C(x) = m^{-1}(x)$ would have a corner at $x=1$ and so isn't differentiable there.

Answer: $C'(1) =$ **DNE**

e. [2 points] On which of the following intervals does $m(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers.

$[-1,2]$ $\qquad$ $[0,5]$ $\qquad$ $\boxed{[3,5]}$ $\qquad$ NONE OF THESE

2. [9 points] Suppose $q(t)$ is a continuous function defined for all real numbers t. The **derivative** and **second derivative** of $q(t)$ are given by

$$q'(t) = te^{t/2}|t-3| \qquad \text{and} \qquad q''(t) = \frac{e^{t/2}(t-3)(t-2)(t+3)}{2|t-3|}.$$

Throughout this problem, you must use calculus to find and justify your answers. Make sure you show enough evidence to justify your conclusions.

a. [5 points] Find the t-coordinates of all local minimum(s) and local maximum(s) of $q(t)$. If there are none of a particular type, write NONE.

Solution: The critical points of q are where $q'(t) = 0$, which occurs at $t = 0$ and $t = 3$. There are no points at which q' DNE.

(checking signs for 1st Derivative Test)	$t < 0$	$0 < x < 3$	$3 < t$
t	$-$	$+$	$+$
$e^{t/2}$	$+$	$+$	$+$
$\lvert t-3\rvert$	$+$	$+$	$+$
$q'(t) = te^{t/2}\lvert t-3\rvert$	$-\cdot+\cdot+=-$	$+\cdot+\cdot+=+$	$+\cdot+\cdot+=+$

This gives the following number line for $q'(t)$:

$q'(t)$	$-\cdot+\cdot+=-$	0	$+\cdot+\cdot+=+$	3	$+\cdot+\cdot+=+$

By the First Derivative Test, $q(t)$ has a local min at $t = 0$. There is no local extremum at $t = 3$.

Answer: Local min(s) at $t =$ 0 and Local max(es) at $t =$ **NONE**

b. [4 points] Find the t-coordinates of all inflection points of $q(t)$, or write NONE if there are none.

Solution: We start by finding any values of t for which $q''(t) = 0$ or q'' DNE, and find $t = -3$, 2, and 3. Now we need to check to see whether the concavity of q changes at these points:

$q''(t)$	$\frac{-\cdot-\cdot-}{+}=-$	-3	$\frac{-\cdot-\cdot+}{+}=+$	2	$\frac{-\cdot+\cdot+}{+}=-$	3	$\frac{+\cdot+\cdot+}{+}=+$

Because the sign of the second derivative of q changes at each of these points, they are all inflection points.

Answer: Inflection point(s) at $t =$ **-3, 2, 3**

3. [8 points] At a certain location in Lake Michigan, scientists are measuring water temperature. Let $W(d)$ be the temperature, in degrees Fahrenheit ($^\circ F$), of the water at a depth of d meters (m). Shown below is a table of values of $W(d)$ and its derivative $W'(d)$, which are both defined and differentiable for all $d \geq 0$.

d	10	18	20	36	78	95
$W(d)$	62	56	55	50	43	41
$W'(d)$	-1.25	-0.60	-0.45	-0.28	-0.15	-0.10

Assume that between each pair of consecutive values of d given in the table, each function $W(d)$ and $W'(d)$ is either always increasing or always decreasing. *Throughout this problem, you do not need to include units or simplify numerical values.*

a. [1 point] Use the table to approximate the value of $W''(19)$.

Answer: $W''(19) \approx$ $\frac{0.15}{2}$

b. [2 points] Write a formula for the linear approximation $L(d)$ of $W(d)$ near $d = 95$.

Answer: $L(d) =$ $41 - 0.1(d - 95)$

c. [1 point] Use your formula from part **b**. to approximate the water temperature, in $^\circ F$, of the water at a depth of 90 meters.

Answer: $41 - 0.1(90 - 95) = 41.5$

d. [1 point] Is your estimate from part **c**. an overestimate, an underestimate, neither, or is there not enough information to decide? Circle your answer.

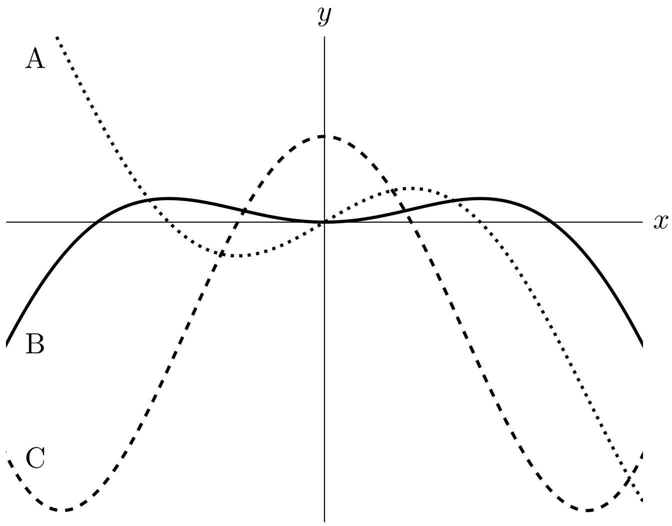
Circle One: OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ NEITHER NOT ENOUGH INFO

e. [3 points] The scientists are taking measurements using an underwater drone. The depth d, in meters, of the drone after t minutes of taking measurements can be modeled by $d = 3\sqrt{t}$. Let $R(t) = W\left(3\sqrt{t}\right)$ be the temperature in $^\circ F$ outside the drone t minutes into the measurements. Write a formula for the linear approximation $K(t)$ of $R(t)$ near $t = 36$.

Solution: We know $K(t) = R(36) + R'(36)(t - 36)$. Now, $R(36) = W\left(3\sqrt{36}\right) = W(18) = 56$. Also, $R'(t) = W'\left(3\sqrt{t}\right) \cdot \frac{3}{2\sqrt{t}}$ by the Chain Rule, so $R'(36) = W'(18) \cdot \frac{3}{12} = -\frac{0.6}{4}$.

Answer: $K(t) =$ $56 - \frac{0.6}{4}(t - 36)$

4. [4 points] Shown below are portions of the graphs of $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $f(x)$: B

$f'(x)$: A

$f''(x)$: C

5. [7 points] The function $p(x)$ is given by the following formula, where c and d are nonzero constants:

$$p(x) = \begin{cases} \frac{1}{3}x^3 - 9x + 1 & x \leq 0 \\ 2^x & 0 < x < 2 \\ c + d(x-2) & x \geq 2. \end{cases}$$

a. [3 points] Find one pair of values for c and d such that $p(x)$ is differentiable at $x = 2$. Show your work.

Solution: For $p(x)$ to be differentiable at $x = 2$, it must first be continuous there, so we need $c + d(2-2) = c$ to equal $2^2 = 4$. Then, we need the slopes on either side of $x = 2$ to be the same. For $0 < x < 2$ we have $p'(x) = \ln(2)2^x$, while for $x > 2$ we have $p'(x) = d$, so we need $d = \ln(2)2^2$.

Answer: $c =$ 4 and $d =$ $4\ln(2)$

b. [4 points] For the values of c and d from part **a.**, find the x-coordinates of all critical points of $p(x)$ or write NONE if there are none. Show your work.

Solution: For $x < 0$ we have $p'(x) = x^2 - 9$. Then, as above, for $0 < x \geq 2$ we have $p'(x) = \ln(2)2^x$, while for $x \geq 2$ we have $p'(x) = d = 4\ln(2)$. Recall that we chose the values of c and d in part **a.** so that $p(x)$ was differentiable at $x = 2$, so we know $p'(2)$ exists. However, we need to consider whether $p'(x)$ exists at $x = 0$: to the left of $x = 0$, the slope is $0^2 - 9 = -9$, while to the right, the slope is $\ln(2)$, and since these are not equal, $p'(x)$ does not exist at $x = 0$, which is therefore a critical point of $p(x)$.

Then, we note that $p'(x)$ is never 0 for $x > 0$, but $x^2 - 9 = 0$ when $x = \pm 3$. However, this formula is only relevant for $x < 0$, so this only gives -3 as an additional critical point.

Answer: Critical point(s) at $x =$ **-3, 0**

6. [6 points] The *Lambda* app team needs to set up an internet connection for their new office. They will utilize both fiber internet and wireless 5G. Due to some serious issues with the building, the team knows that 10% of the data transferred via fiber will be lost, and 30% of the data transferred via the wireless connection will be lost. Assume throughout this problem that, each month, the company loses exactly 15 gigabytes (GB) of transferred data.

a. [2 points] If each month, the team transfers F GB of data via fiber and W GB via wireless, write a formula for F in terms of W.

Solution: We know that 10% of of F and 30% of W will be lost, and that the company will lose 15 GB a month, so $15 = 0.1F + 0.3W$, or $\dfrac{15 - 0.3W}{0.1} = F$.

Answer: $F = \dfrac{15 - 0.3W}{0.1}$

However, the team also wants to consider the monthly energy consumption of the methods, which, in gigajoules (GJ), is given by

$$(W + 1)^4 + (F + 4)^2.$$

b. [1 point] Write a formula for the energy $\mathcal{E}(W)$, in GJ, as a function of W only. *Your formula should not include the letter F.*

Solution: We substitute the answer to part **a.** into the given formula for energy:

Answer: $\mathcal{E}(W) = (W + 1)^4 + \left(\dfrac{15 - 0.3W}{0.1} + 4\right)^2$

c. [3 points] Additionally, the team wants to ensure that $2W \geq F$. Including this additional constraint, what is the domain of the function $\mathcal{E}(W)$ in the context of this problem?

Solution: We must have $W > 0$, as well as $F = \dfrac{15 - 0.3W}{0.1} > 0$, or $W < \dfrac{15}{0.3} = 50$. Then we are also told $2W \geq F$, so

$$\begin{aligned} 2W &\geq \frac{15 - 0.3W}{0.1} \\ 0.2W &\geq 15 - 0.3W \\ 0.5W &\geq 15 \\ W &\geq 30 \end{aligned}$$

Answer: $[30, 50)$

7. [8 points] The *Lambda* app team also needs to write some new code and so wants to hire one or more temporary programmers. They will pay \$300 per day to hire an experienced programmer for x days, and \$100 per day to hire a beginner programmer for y days. They want to spend \$900 in total, so

$$9 = 3x + y.$$

However, the team expects that the programmer(s) will drink of a lot of coffee. In particular, they believe the cost C, in dollars, of the coffee they will need can be modeled by

$$C = y - x^3 + 3x^2 + 3x - 8.$$

What value(s) of x will minimize the cost of the coffee? Use calculus to find and justify your answers, and be sure to show enough evidence that the value(s) you find do in fact minimize the cost of the coffee.

Solution: First, we solve for $y = 9 - 3x$ and substitute this into C to get

$$C(x) = (9 - 3x) - x^3 + 3x^2 + 3x - 8 = 1 + 3x^2 - x^3.$$

Then we find all critical points of $C(x)$ using

$$C'(x) = 6x - 3x^2 = 3x(2 - x)$$

which is never undefined and zero at 0 and 2, so our critical points are 0 and 2. We should make sure they are both relevant to our domain: in this case $x \geq 0$ and $y = 9 - 3x \geq 0$, so $x \leq 3$, so our domain is $[0, 3]$ and both critical points are relevant. Now, we find the value of $C(x)$ at both endpoints and the remaining critical point:

x	$C(x)$
0	1
2	5
3	1

We see that both 0 and 3 lead to the minimum cost.

Answer: value(s) of x that minimize the cost of coffee: $0, 3$

8. [7 points] The function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \dfrac{x}{x^2+1} & x \le 0 \\ ? & x > 0. \end{cases}$$

Note that the formula for $f(x)$ for $x > 0$ is unknown. However, it is known that $f(x)$ is differentiable at each point in its domain $(-\infty, \infty)$, and that $f'(x) > 0$ for all $x \ge 0$.

a. [4 points] Find the x-coordinates of all global minimum(s) and global maximum(s) of $f(x)$ **on the interval** $(-\infty, 0]$. If there are none of a particular type, write NONE. Use calculus to find your answers, and make sure that you show enough evidence to justify your conclusions.

Solution: First, using the quotient rule, we find that $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ for $x < 0$. There are no values of x where this is undefined, so to find critical points, we set $f'(x)$ equal to 0 to find that $x = \pm 1$. However, this formula is only relevant for $x < 0$ so our only critical point for $x < 0$ is $x = -1$.
Then, we see that

x	$f(x)$
0	0
-1	$-\frac{1}{2}$
$\lim\limits_{x\to-\infty} f(x)$	0

Then we have a global min at $x = -1$, and a global max at $x = 0$.

Answer: Global min(s) at $x =$ -1

Answer: Global max(es) at $x =$ 0

b. [3 points] For each question below, circle all correct answers. No justification is needed.

At which of the following value(s) of x does $f(x)$ attain a global minimum **on the interval** $[-2, 2]$?

$x = -2$ $\quad$ $\boxed{x = -1}$ $\quad$ $x = 0$ $\quad$ $x = 1$ $\quad$ $x = 2$ $\quad$ NONE OF THESE

At which of the following value(s) of x does $f(x)$ attain a global maximum **on the interval** $[-2, 2]$?

$x = -2$ $\quad$ $x = -1$ $\quad$ $x = 0$ $\quad$ $x = 1$ $\quad$ $\boxed{x = 2}$ $\quad$ NONE OF THESE

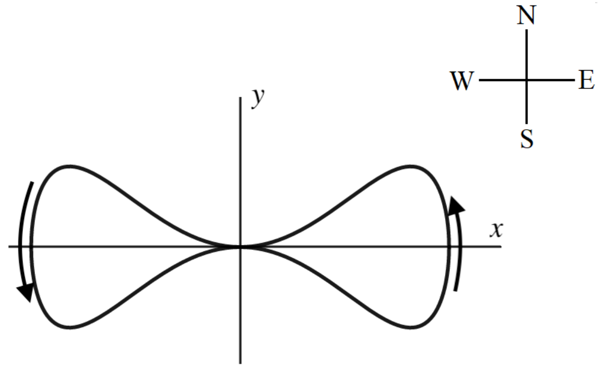
9. [7 points]

You are on a hiking trip, following the path modeled by the curve $\mathcal{B}$ defined by the equation

$$y^2 = x^4(1-x^2).$$

Note that

$$\frac{dy}{dx} = \frac{x^3(2-3x^2)}{y}.$$



The graph of $\mathcal{B}$ is shown to the right. You begin your hike at $(0,0)$, then:

- travel East and around the loop on the right as shown by the arrow, returning to $(0,0)$, then
- travel West and around the loop on the left as shown by the arrow, returning to $(0,0)$.

a. [5 points] Using calculus, find the coordinates of all the other points (x,y) on your path (that is, other than $(0,0)$), where you travel directly East or directly West. Show your work.
Note that you can use the graph to determine how many points you are looking for.

Solution: We look for where the numerator of $\frac{dy}{dx}$ is 0, i.e. $x^3(2-3x^2)=0$. We ignore the solution $(0,0)$, so we need $2-3x^2=0$ or $x=\pm\sqrt{\frac{2}{3}}$. Then

$$y^2 = \left(\pm\sqrt{\frac{2}{3}}\right)^4\left(1-\left(\pm\sqrt{\frac{2}{3}}\right)^2\right)$$

$$y^2 = \frac{4}{9}\left(1-\frac{2}{3}\right) = \frac{4}{9}\cdot\frac{1}{3} = \frac{4}{27}$$

so $y=\pm\frac{2}{3\sqrt{3}}$. We can see which direction we are traveling at these points from the graph.

Answer: travel East at $\left(\sqrt{\frac{2}{3}}, -\frac{2}{3\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, -\frac{2}{3\sqrt{3}}\right)$

Answer: travel West at $\left(\sqrt{\frac{2}{3}}, \frac{2}{3\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, \frac{2}{3\sqrt{3}}\right)$

b. [2 points] Using calculus, find the coordinates of all the points (x,y) on your path where you travel directly North or directly South. Note that, as shown by the graph, $(0,0)$ is not one of these points. Show your work.

Solution: We look for where the denominator of $\frac{dy}{dx}$ is 0, i.e. $y=0$. Then $0=x^4(1-x^2)$ so either $x=0$ or $x=\pm1$. We ignore $(0,0)$, so the points are $(1,0)$ and $(-1,0)$.

Answer: travel North at $(1,0)$

Answer: travel South at $(-1,0)$

10. [7 points] The curve $\mathcal{C}$ is given by the equation $e^{\cos\left(x^2-y^2\right)} = ex$.

a. [2 points] Which of the following points (x, y) lie on the curve $\mathcal{C}$? Circle all correct answers.

$\boxed{(1,1)}$ $(-1,1)$ $\boxed{(1,-1)}$ $(0,\sqrt{2}\,)$ NONE OF THESE

b. [5 points] Compute $\frac{dy}{dx}$. Show every step of your work and circle your final answer.

Solution: Implicitly differentiating, we get: $e^{\cos(x^2-y^2)} \cdot \left(-\sin(x^2-y^2)\right) \cdot \left(2x - 2y\frac{dy}{dx}\right) = e.$

Now we solve for $\frac{dy}{dx}$:

$$2x - 2y\frac{dy}{dx} = \frac{e}{e^{\cos(x^2-y^2)} \cdot \left(-\sin(x^2-y^2)\right)},$$

$$-2y\frac{dy}{dx} = \frac{e}{e^{\cos(x^2-y^2)} \cdot \left(-\sin(x^2-y^2)\right)} - 2x,$$

$$\frac{dy}{dx} = \frac{e}{-2ye^{\cos(x^2-y^2)} \cdot \left(-\sin(x^2-y^2)\right)} - \frac{2x}{-2y} = \frac{e}{2ye^{\cos(x^2-y^2)}\sin(x^2-y^2)} + \frac{x}{y}.$$

Or, one can distribute the left-hand side first, to obtain the equivalent answer

$$\frac{dy}{dx} = \frac{2xe^{\cos(x^2-y^2)} \cdot \sin(x^2-y^2) + e}{2ye^{\cos(x^2-y^2)} \cdot \sin(x^2-y^2)}.$$

11. [8 points]

Suppose $h(x)$ is a function such that $h(x)$ has **exactly three** critical points. Assume that both $h(x)$ and $h'(x)$ are differentiable on $(-\infty, \infty)$. A table of values is given to the right.

x	0	3	5	7
$h(x)$	2	?	4	4
$h'(x)$	-1	0	0	?

a. [2 points] Note that $h(x)$ satisfies the hypotheses of the Mean Value Theorem on $[5, 7]$. Briefly explain why the conclusion of this theorem implies that one of the three critical points of $h(x)$ must be in the interval $5 < x < 7$.

Solution: The conclusion of the Mean Value Theorem applied to $h(x)$ on $[5, 7]$ states that there exists a point c with $5 < c < 7$ such that $h'(c) = \frac{h(7)-h(5)}{7-5} = \frac{4-4}{7-5} = 0$, making c a critical point of $h(x)$ in the interval $5 < x < 7$.

In the parts below, circle all correct answers. No justification is needed.

b. [2 points] On which of the following intervals *must* $h(x)$ be increasing on the entire interval?

$(0,3)$ $\boxed{(3,5)}$ $(5,6)$ $(6,7)$ NONE OF THESE

c. [2 points] Which of the following *could* be the x-coordinate of a global minimum of $h(x)$ on $(-\infty, \infty)$?

$x = 0$ $\boxed{x = 3}$ $x = 5$ $\boxed{x = 6}$ NONE OF THESE

d. [2 points] Also suppose that $h(x)$ is concave down on $(-\infty, 0)$. Which of the following *could* be the value of $h(-2)$?

1 2 $\boxed{3}$ 4 5 NONE OF THESE