Math 115 — Final Exam — December 14, 2022

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 9 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a single $3'' \times 5''$ notecard.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	13	
2	11	
3	9	
4	14	
5	11	

Problem	Points	Score
6	11	
7	9	
8	10	
9	12	
Total	100	

1. [13 points] Given below is a table of values for a function $h(x)$ and its derivative $h'(x)$. The functions $h(x)$, $h'(x)$, $h''(x)$, and $h'''(x)$ are all defined and continuous on $(-\infty, \infty)$.

x	-6	-4	-2	0	2	4	6
$h(x)$	2	-0.5	-2	-3	1	4	3
$h'(x)$	0	-4	-1	0	3	0	-2

Assume that between consecutive values of x given in the table above, $h(x)$ is either **always increasing** or **always decreasing**.

In **a.–c.**, find the numerical value **exactly**, or write NEI if there is not enough information provided to do so. *You do not need to show work on this page, but limited partial credit may be awarded for work shown.*

a. [2 points] Find the average rate of change of $h(x)$ from $x = -6$ to $x = -2$.

Answer: $\dfrac{-2-2}{-2-(-6)} = -1$

b. [2 points] If the average value of $h'''(x)$ on the interval $[-6, 0]$ is 2, find $5 \cdot \displaystyle\int_{-6}^{0} \big(1 + h'''(x)\big)\ dx$.

Solution:
$\dfrac{1}{6}\displaystyle\int_{-6}^{0} h'''(x)\ dx = 2$, so $\displaystyle\int_{-6}^{0} h'''(x)\ dx = 12$. Then $5 \cdot \displaystyle\int_{-6}^{0} \big(1 + h'''(x)\big)\ dx = 5 \cdot 6 + 5 \cdot 12$

Answer: 90

c. [3 points] Find $\displaystyle\int_{-4}^{-2} \big(2h'(x) + x\big)\ dx$.

Solution: $= 2\displaystyle\int_{-4}^{-2} h'(x)\ dx + \int_{-4}^{-2} x\ dx = 2(h(-2) - h(4)) + \left.\frac{x^2}{2}\right|_{-2}^{-4} = 2(-2-(-0.5)) + \frac{4}{2} - \frac{16}{2}$

Answer: -9

d. [2 points] Find an equation for the tangent line to the graph of $h(x)$ at $x = 6$.

Answer: $y - 3 = -2(x-6)$ **or** $y = -2x + 15$

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$h(x)$	2	-0.5	-2	-3	1	4	3
$h'(x)$	0	-4	-1	0	3	0	-2

Assume that between consecutive values of x given in the table above, $h(x)$ is either **always increasing** or **always decreasing**.

e. [2 points] Use a left Riemann sum with three equal subdivisions to estimate $\int_{-6}^{6} h(x)\,dx$. Write out all the terms in your sum, which you do not need to simplify.

Solution:
$4 \cdot h(-6) + 4 \cdot h(-2) + 4 \cdot h(2)$
$= 4 \cdot 2 \quad + 4 \cdot (-2) \quad + 4 \cdot (1)$

f. [2 points] Fill in each blank below with one of the following:

$\boxed{\leq}$, $\boxed{\geq}$, $\boxed{=}$ or $\boxed{\text{NEI}}$

where NEI means there is not enough information to decide. You need not justify your answers.

i. $\int_{-6}^{0} h(x)\,dx$ ___$\leq$___ $2h(-6) + 2h(-4) + 2h(-2)$.

ii. $\int_{0}^{6} h(x)\,dx$ ___NEI___ $2h(0) + 2h(2) + 2h(4)$.

2. [11 points] The following parts are unrelated.

a. [3 points] Which of the following limits are equal to 0? Circle all correct answers.

i. $\lim_{x\to\infty} \frac{3x^2 - x + 5}{1 - x^2}$

ii. $\boxed{\lim_{x\to\infty} \frac{x^3}{1 - x^4}}$

iii. $\boxed{\lim_{x\to\infty} \frac{x^2}{e^{2x}}}$

iv. $\lim_{x\to-\infty} \frac{x^2}{e^{2x}}$

v. $\lim_{h\to 0} \frac{\sin(h) - \sin(0)}{h}$

vi. $\boxed{\lim_{h\to 0} \frac{\cos(h) - \cos(0)}{h}}$

vii. NONE OF THESE

b. [2 points] If $k(x)$ is an **odd** function that is differentiable on $(-\infty, \infty)$, which of the following must be true? Circle all correct answers.

i. $k'(x)$ is an odd function

ii. $\boxed{k(0) = 0}$

iii. $\boxed{\int_{-2}^{2} k(x)\, dx = 0}$

iv. $\int_{-3}^{1} k(x)\, dx = \int_{-1}^{3} k(x)\, dx$

v. NONE OF THESE

c. [2 points] Which of the following is a formula for the linear approximation to xe^{2x} at $x = 1$? Circle the one correct answer.

i. $2e^2x - e^2$

ii. $e^2 + (2xe^{2x} + e^{2x})(x - 1)$

iii. $\boxed{3e^2(x - 1) + e^2}$

iv. $e^2 + e^2(x - 1)$

v. $3e^2x + e^2$

vi. NONE OF THESE

d. [4 points] A company's maximum profit is earned when it produces $q = 8$ units of their product. If its marginal revenue function is $MR(q) = 3$, which of the following could be true? Circle all correct answers.

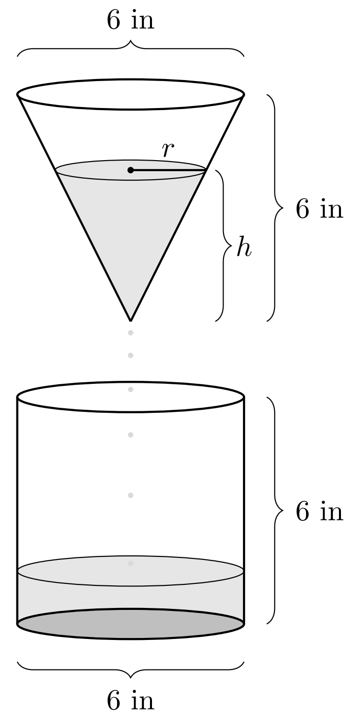
i. the company's cost function is $C(q) = \frac{q^2}{2} - 5q$, and they can produce at most 12 units of their product

ii. the company's cost function is $C(q) = 2q$, and they can produce at most 8 units of their product

i. the company's marginal cost function is $MC(q) = 4$, and they can produce at most 8 units of their product

iv. the company's marginal cost function is $MC(q) = \sqrt{q+1}$, and they can produce at most 15 units of their product

ii. NONE OF THESE



3. [9 points] Coffee is draining from a cone filter into a cylindrical pot, as shown in the figure to the right. The height and diameter of both the filter and the coffee pot are 6 inches.

Let r be the radius of the circular surface and h be the height of the coffee remaining in the filter, both measured in inches. Note that the shape of the filter implies that $r = \frac{h}{2}$.

Recall that the volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$, while the volume of a circular cylinder with radius R and height H is $\pi R^2 H$.

At the moment in time when the height of the coffee in the filter is 5 inches, the coffee is draining from the filter at a rate of 10 cubic inches per minute.

a. [5 points] At what rate is the height of the coffee in the filter decreasing at that moment? *Include units.*

Solution: Let V be the volume of coffee in the filter, so

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$$

Differentiating with respect to t, we get

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}.$$

Since $\frac{dV}{dt} = -10$ and $h = 5$, this means

$$-10 = \frac{1}{4}\pi(5)^2\frac{dh}{dt}, \quad \text{so} \quad \frac{dh}{dt} = \frac{-40}{25\pi} = \frac{-8}{5\pi}.$$

Thus the height of the coffee is decreasing at a rate of $\frac{8}{5\pi}$ inches per minute.

Answer: $\frac{8}{5\pi}$ **inches per minute**

b. [4 points] At what rate is the height of the coffee in the pot increasing at that moment? *Include units.*

Solution: Let V be the volume and H the height of coffee in the pot, so

$$V = \pi \cdot 3^2 H = 9\pi H.$$

Differentiating with respect to t, we get

$$\frac{dV}{dt} = 9\pi \frac{dH}{dt},$$

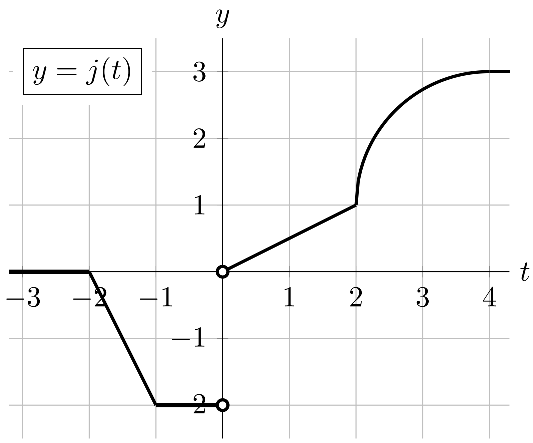
so

$$\frac{dH}{dt} = \frac{10}{9\pi}.$$

Thus the height of coffee in the pot is increasing at a rate of $\frac{10}{9\pi}$ inches per minute.

Answer: $\frac{10}{9\pi}$ **inches per minute**

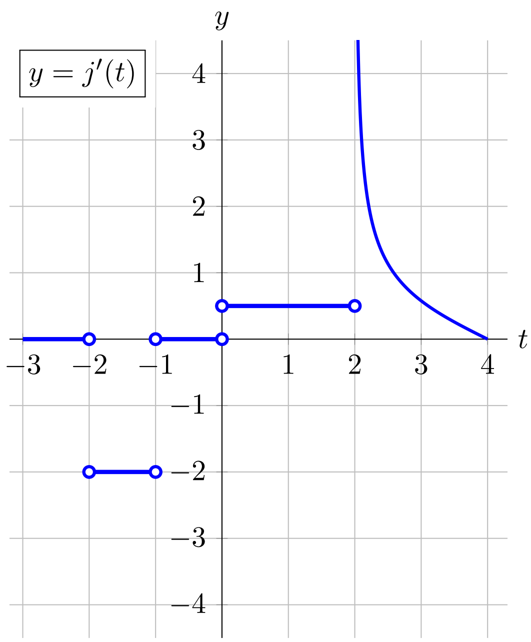
4. [14 points]
A portion of the graph of the function $j(t)$ is shown to the right. Note that, on the interval $2 \le t \le 4$, the graph consists of a quarter of a circle that is centered at the point $(4, 1)$.



a. [6 points]

On the axes to the right, sketch a detailed graph of $j'(t)$, the derivative of $j(t)$, for $-3 \le t \le 4$. Make sure that the following are clear from your graph:

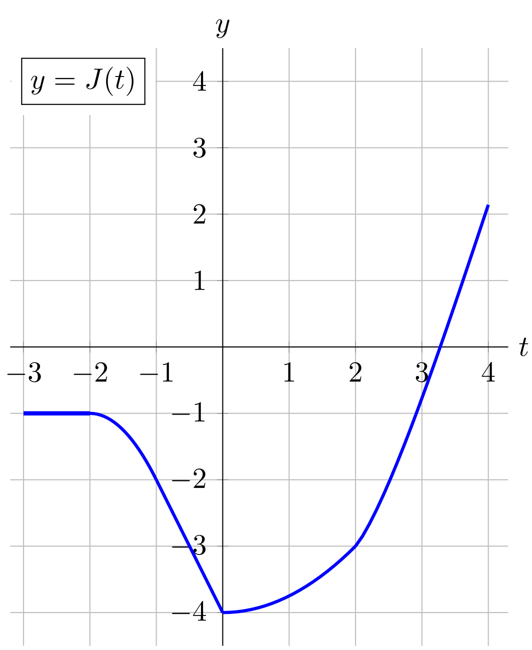
- where $j'(t)$ is undefined
- any vertical asymptotes of $j'(t)$
- where $j'(t)$ is zero, positive, and negative
- where $j'(t)$ is increasing, decreasing, and constant



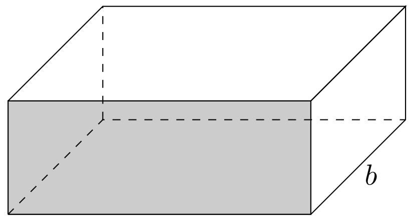
b. [8 points]

Let $J(t)$ be a continuous antiderivative of $j(t)$ with $J(-1) = -2$. On the axes to the right, sketch a detailed graph of $J(t)$ for $-3 \le t \le 4$. Make sure that the following are clear from your graph:

- where $J(t)$ is and is not differentiable
- the values of $J(t)$ at $t = -3, -2, -1, 0, 2,$ and 4
- where $J(t)$ is increasing, decreasing, and constant
- where $J(t)$ is linear (with correct slope)
- the concavity of $J(t)$



5. [11 points] A museum is building a specialized box to display a new exhibit. The box needs to have a volume of 20 cubic feet and a height of 2 feet. The front of the box, which is shaded in the diagram below, will be made of glass, which costs \$4 per square foot. The top, sides, back, and bottom of the box will be made of a metal that costs \$1 per square foot. Let a and b be the length and width, in feet, of the box, as shown below.



What values of a and b will minimize the cost of the box, and what will the cost be in that case? Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact minimize the cost of the box.

Solution:
The cost for the box is

$$C = 4 \cdot 2a + ab + ab + 2a + 2b + 2b$$
$$= 10a + 2ab + 4b.$$

Solving for a:

$$20 = 2ab \text{ , so } a = \frac{10}{b}$$

Therefore the cost function is

$$C(b) = \frac{10 \cdot 10}{b} + \frac{2 \cdot 10b}{b} + 4b$$
$$= \frac{100}{b} + 20 + 4b$$
$$C'(b) = -\frac{100}{b^2} + 4$$

is zero when $b^2 = 25$ or $b = 5$.
Now $C(5) = 60$ while

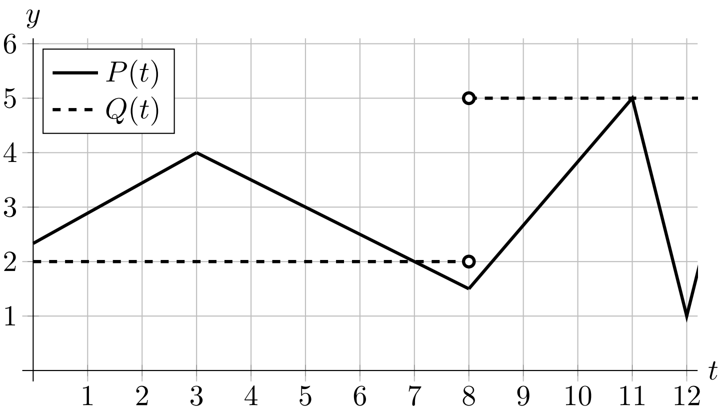
$$\lim_{b \to 0^+} C(a) = \infty \text{ and } \lim_{b \to \infty} C(a) = \infty,$$

so $b = 5$ is the **global max**, with $a = \frac{10}{5} = 2$.

Answer: the cost is minimized when $a =$ 2 and $b =$ 5

and the minimum cost is \$ 60

6. [11 points] After selling *Lambda Inc.*, the team is doing a volunteer data analysis of a pond near Ann Arbor, in which the amount of water changes over time due to various factors such as streams, rain, and evaporation. Considering all such factors combined, let $P(t)$ be the rate of water entering the pond, and let $Q(t)$ be the rate of water leaving the pond, both measured in thousands of tons per hour, t hours after noon on a particular day. (That is, $t = 0$ is noon, $t = 1$ is 1 pm, etc.). The graphs of $P(t)$ and $Q(t)$ are given below.



a. [2 points] At which of the following times t is the amount of water in the pond decreasing? Circle all correct answers.

$t = 2$ $\qquad$ $t = 4$ $\qquad$ $\boxed{t = 9}$ $\qquad$ $\boxed{t = 11.5}$ $\qquad$ NONE OF THESE

b. [2 points] At what time(s) t for $0 \le t \le 12$ is the amount of water in the pond changing the fastest?

Answer: $t =$ 12

c. [2 points] At what time(s) t for $0 \le t \le 12$ does the pond have the greatest amount of water?

Answer: $t =$ 7

In parts **d.** and **e.** below, give your answers in terms of $P(t)$, $Q(t)$, their derivatives, and/or definite integrals. Do not attempt to numerically evaluate any expressions in your answers.

d. [2 points] Write a single expression for the total amount of water that enters the pond from 5 pm to 7 pm.

Answer: $\int_5^7 P(t)\,dt$

e. [3 points] Write a single equation representing the following statement:

The total change in the amount of water in the pond from noon to midnight is zero.

Answer: $\int_0^{12} P(t) - Q(t)\,dt = 0$ **or** $\int_0^{12} P(t)\,dt = \int_0^{12} Q(t)\,dt$

7. [9 points] Consider the family of functions

$$m(x) = x + \frac{c^2}{x}$$

defined for $x > 0$, where c is a positive constant.

Throughout this problem, use calculus to find your answers, show all your work, and be sure to show enough evidence to justify your conclusions.

a. [2 points] Any function in this family has only one critical point on its domain $x > 0$. In terms of c, what is the x-coordinate of this critical point?

Solution: $m'(x) = 1 - \frac{c^2}{x^2}$.
$m'(x)$ DNE at $x = 0$ which is **not** in the domain.
$m'(x) = 0$ at $x = c$ which **is** in the domain.

Answer: c **or** $x = c$

b. [3 points] Is the critical point a local minimum, a local maximum, neither, or is there not enough information to decide? Circle your answer below.

Solution:
For the 1^{st} *derivative test:* $m'(\frac{c}{2}) = 1 - 4\frac{c^2}{c^2} = -3 < 0$ and $m'(2c) = 1 - \frac{c^2}{4c^2} = \frac{3}{4} > 0$. Therefore $x = c$ is a **local min**.
For the 2^{nd} *derivative test:* $m''(x) = 0 + 2\frac{c^2}{x^3}$ and $m''(c) = \frac{2}{c} > 0$. Therefore $x = c$ is a **local min**.

Answer: $\boxed{\text{local min}}$ local max neither not enough info

c. [2 points] Find the x-coordinates of all inflection points of $m(x)$, or if there are none, write NONE.

Solution: $m''(x) = 0 + 2\frac{c^2}{x^3}$ which is defined everywhere in the domain and not equal to zero on the domain. Therefore $m(x)$ has no inflection points.

Answer: Inflection point(s) at $x =$ **NONE**

d. [2 points] Find the value for c such that $m(x) = 10$ at its critical point.

Solution: The value at $x = c$ is $m(c) = c + \frac{c^2}{c} = 2c$. If it is equal to 10, then $2c = 10$, or $c = 5$.

Answer: $c =$ 5

8. [10 points] Scientists are continuing their study of water temperature in Lake Michigan using their underwater drone. Assume the following functions $W(p)$ and $T(p)$ are invertible and differentiable:

- Let $W(p)$ be the temperature, in degrees Celsius (°C), of the water at a depth of p feet.
- Let $T(p)$ be the time, in minutes, that it takes for the drone to descend to a depth of p feet.

a. [2 points] Write a single equation representing the following statement in terms of the functions W, T, and/or their inverses:

It takes the drone 3 minutes to reach water with a temperature of 5.7°C.

Answer: $T\left(W^{-1}(5.7)\right) = 3$ **or** $W\left(T^{-1}(3)\right) = 5.7$

b. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$W'(50) = -0.2.$$

Compared to the water at a depth of 50 feet, the water at a depth of 53 feet...

Solution: ...is about 0.6°C colder.

c. [3 points] Use a complete sentence to give a practical interpretation of the equation

$$\int_{50}^{65} T'(p)\, dp = 1.$$

Solution: It takes the drone one minute to descend from a depth of 50 ft to a depth of 65 ft.

d. [2 points] Which of the following expressions gives the average temperature, in °C, of the water outside of the drone during the first 5 minutes of its descent? Circle the one correct answer.

i. $\frac{1}{5}\int_0^5 W(p)\, dp$

$\boxed{\text{ii.}}$ $\frac{1}{5}\int_0^5 W(T^{-1}(t))\, dt$

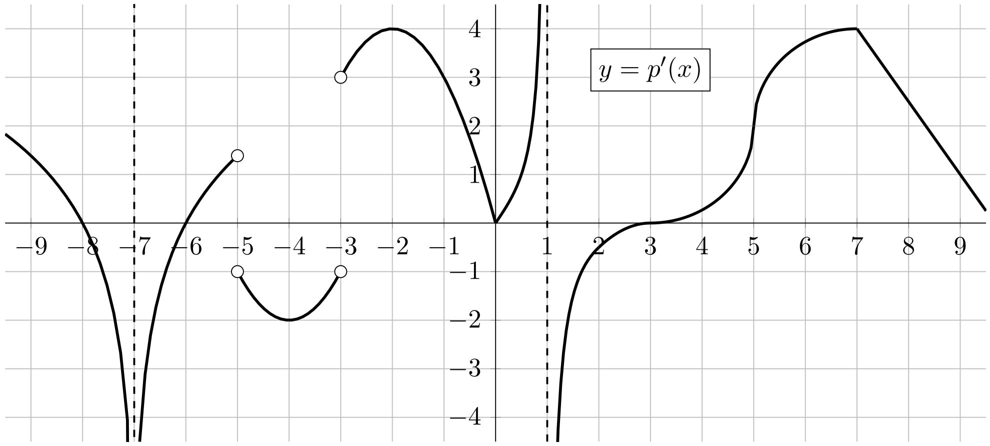
iii. $\frac{1}{5}\int_0^5 W'(T(t)) \cdot T'(t)\, dt$

iv. $\dfrac{W(T^{-1}(5)) - W(T^{-1}(0))}{5}$

v. $\dfrac{W(T(5)) + W(T(0))}{2}$

vi. $\dfrac{T(W^{-1}(5)) - T(W^{-1}(0))}{5}$

9. [12 points] A function $p(x)$ is continuous on $(-\infty, \infty)$. Part of its **derivative** $p'(x)$ is shown below.



a. [2 points] At which of the following x-value(s) does $p(x)$ have a critical point?

$\boxed{-8}$ $\qquad$ $\boxed{-3}$ $\qquad$ -2 $\qquad$ $\boxed{1}$ $\qquad$ 7

b. [2 points] At which of the following x-value(s) does $p(x)$ have a local minimum?

-8 $\qquad$ -4 $\qquad$ $\boxed{-3}$ $\qquad$ 0 $\qquad$ $\boxed{3}$

c. [2 points] On which of the following interval(s) is $p(x)$ increasing on the entire interval?

$(-4, -3)$ $\qquad$ $\boxed{(-3, 0)}$ $\qquad$ $(1, 2)$ $\qquad$ $\boxed{(3, 7)}$

d. [2 points] On which of the following interval(s) is $p(x)$ concave up on the entire interval?

$(-5, -3)$ $\qquad$ $\boxed{(0, 1)}$ $\qquad$ $\boxed{(3, 7)}$ $\qquad$ $(7, 9)$

e. [2 points] On which of the following interval(s) is the product $p'(x) \cdot p''(x)$ negative on the entire interval?

$(-5, -4)$ $\qquad$ $\boxed{(-2, 0)}$ $\qquad$ $\boxed{(1, 3)}$ $\qquad$ $(3, 5)$

f. [2 points] At which of the following x-value(s) does $p(x)$ have both a local extremum and an inflection point?

-7 $\qquad$ $\boxed{-5}$ $\qquad$ -3 $\qquad$ 0 $\qquad$ 3 $\qquad$ NONE OF THESE