Math 115 — First Midterm — October 2, 2023

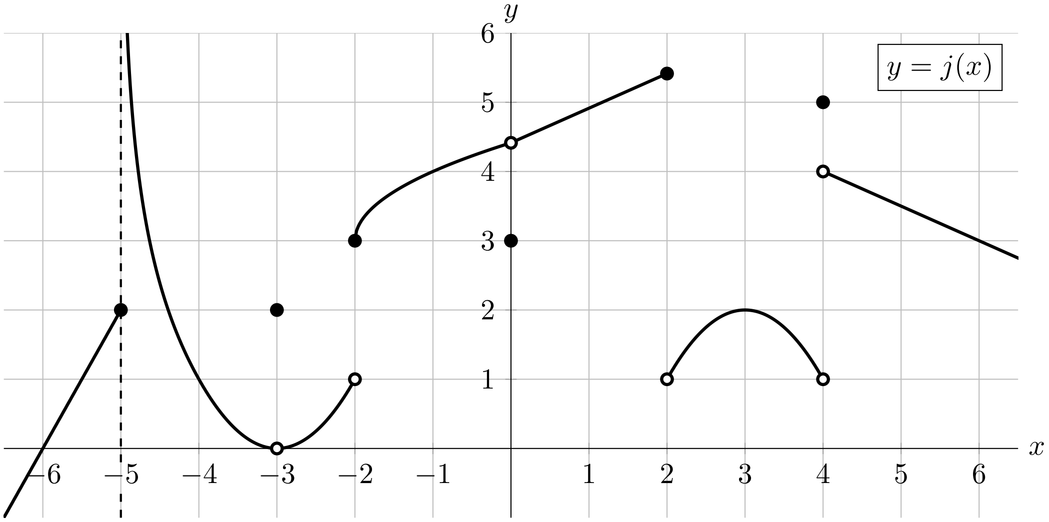
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 8 pages including this cover. There are 8 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	11	
2	9	
3	4	
4	5	
5	7	

Problem	Points	Score
6	8	
7	8	
8	8	
Total	60	

1. [11 points] Below is a portion of the graph of the function $j(x)$. Note that $j(x)$ has a vertical asymptote at $x = -5$, and is linear on the intervals $(-6, -5)$, $(0, 2)$, and $(4, 6)$.



a. [1 point] At which of the following values of x is the function $j(x)$ continuous? Circle all correct answers.

$x = -3$ $x = -2$ [✓ $x = 3$] $x = 4$ NONE OF THESE

b. [6 points] Find the **exact** numerical value of each expression below, if possible. For any values that do not exist, including if they are limits that diverge to ±∞, write DNE.

i. $\lim_{x \to 3} j(x) =$ 2

ii. $\lim_{x \to -3} j(x) =$ 0

iii. $\lim_{x \to 4} j(x) =$ **DNE**

iv. $\lim_{x \to 0} \frac{j(5 + x) - j(5)}{x} =$ $-1/2$

v. $\lim_{x \to 2^+} j(x) =$ 1

vi. $\lim_{x \to -5^+} \frac{1}{j(x)} =$ 0

c. [2 points] Consider the function $k(x) = 2 \cdot j\left(\frac{1}{2}(x - 9)\right) + 1$. Which of the following must be a vertical asymptote of $k(x)$? Circle the one correct answer.

$x = -9$ $x = -5$ $x = -3$ [✓ $x = -1$] $x = 1$

d. [2 points] Given that $j'(-4) = -2$, find an equation of the line tangent to the graph of $j(x)$ at the point $(-4, 1)$.

Solution: An equation of the line tangent to the graph of $j(x)$ at $x = -4$ is given by

$$y - j(-4) = j'(-4)(x + 4).$$

Subbing in $j(-4) = 1$ and $j'(-4) = -2$, we get the equation $y - 1 = -2(x + 4)$.

Answer: $y = 1 - 2(x + 4)$, **or** $y = -2x - 7$

2. [9 points] Your construction company is building an apartment complex with a fixed number of individual apartment units that are all the same size.

- Let $C(a)$ be the cost, in millions of dollars, to construct the apartment complex when each unit has a square feet of space.
- Let $A(u)$ be the size of each apartment unit, in square feet, if u thousand pounds of bricks are used in the construction of the apartment complex.

Assume the functions $C(a)$ and $A(u)$ are invertible and differentiable.

a. [2 points] Fill in the blanks with appropriate *numbers <u>and units</u>* to give a practical interpretation of the equation $A^{-1}(508) = 68$.

In order to build the apartment complex so that each unit has a size of 508 *square feet*,

you will need to use 68,000 *pounds of* *bricks.*

b. [2 points] Write an expression involving A, C, and/or their inverses that represents the following statement:

If the apartment building uses 72,000 pounds of bricks, then it costs 2.8 million dollars to construct the building.

Answer: $C(A(72)) = 2.8$

c. [2 points] Complete the following sentence to give a practical interpretation to the equation

$$A'(73) = 2.$$

If 73,500 pounds of bricks were used to construct the apartment building rather than 73,000, then . . .

Solution: . . . each apartment unit would be about 1 square foot larger.

d. [3 points] Circle the one statement below that is <u>best</u> supported by the equation

$$(C^{-1})'(290) = 3.$$

i. If the amount spent constructing the building is increased to \$310 million from \$290 million, then the size of each apartment increases by 60 square feet.

ii. If the amount of floor space in each apartment is 300 square feet rather than 290 square feet, then the cost to construct the building increases by about 30 million dollars.

[✓ iii.] If the cost of constructing the apartment complex must be cut from \$295 million to \$290 million, then each unit will have to decrease in size by about 15 square feet.

iv. To increase the floor space in each apartment by about 3 square feet, the amount spent in construction needs to be increased by \$1 million dollars.

3. [4 points] Find positive constants a, b, and c such that the function

$$f(x) \;=\; \begin{cases} \ln(ce-x) & x \le 0 \\ \dfrac{ax^3+\pi}{x^b+1} & x > 0 \end{cases}$$

is continuous and satisfies $\lim\limits_{x\to\infty} f(x) = 4$. Show your work, and write your answers in exact form.

Solution: If $b > 3$ then $\lim\limits_{x\to\infty} f(x) = 0$, and if $b < 3$ then $\lim\limits_{x\to\infty} f(x) = \infty$; therefore, since $\lim\limits_{x\to\infty} f(x) = 4$, we know $b = 3$. Then, since $b = 3$, we know that

$$4 \;=\; \lim_{x\to\infty} f(x) \;=\; \lim_{x\to\infty} \frac{ax^3+\pi}{x^3+1} \;=\; a.$$

Finally, since $f(x)$ is continuous at 0, the two parts of the piecewise definition of $f(x)$ must agree at $x = 0$. Plugging $x = 0$ into both pieces, we get

$$\ln(ce) \;=\; \ln(c)+1 \;=\; \pi,$$

so $c = e^{\pi-1}$.

Answers: $a =$ 4 $b =$ 3 $c =$ $e^{\pi-1}$

4. [5 points] Let

$$Q(w) = w^w + \cos(6w-1).$$

Use the limit definition of the derivative to write an explicit expression for $Q'(3)$. *Your answer should not involve the letter Q. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Solution:

$$\begin{aligned} Q'(3) \;&=\; \lim_{h\to 0} \frac{Q(3+h)-Q(3)}{h} \\ &=\; \lim_{h\to 0} \frac{(3+h)^{3+h} + \cos\big(6(3+h)-1\big) - \big[3^3 + \cos(6\cdot 3-1)\big]}{h} \end{aligned}$$

Answer: $Q'(3) =$ $\boxed{\lim\limits_{h\to 0} \dfrac{(3+h)^{3+h} + \cos\big(6(3+h)-1\big) - \big[3^3 + \cos(6\cdot 3-1)\big]}{h}}$

5. [7 points] Since the start of spring, bird enthusiasts Charlie and Parker have been seeing, and feeding, more and more birds in their backyard. Eventually they decide to model the number of birds they see and the amount of birdseed they go through using increasing functions.

Let $B(t)$ be the number of birds they see on the t^{th} day after the start of spring. They start recording values on the third day of spring, and their initial data appear in the table to the right.

t	3	4
$B(t)$	16	20

a. [1 point] Charlie thinks $B(t)$ should be a linear function. Find an expression for $B(t)$ if it is a linear function.

Answer: $B(t) =$ $4(t-3)+16$, **or** $4t+4$

b. [2 points] Parker thinks $B(t)$ should be exponential. Find an expression for $B(t)$ if it is an exponential function.

Solution: If $B(t)$ is exponential, then it has the form $B(t) = B_0a^t$. Thus $16 = B_0a^3$ and $20 = B_0a^4$. Dividing these equations gives

$$a = \frac{20}{16} = 5/4, \qquad \text{so} \qquad B_0 = \frac{16}{(5/4)^3}.$$

Answer: $B(t) =$ $\dfrac{16}{(5/4)^3} \cdot (5/4)^t$

After arguing whether $B(t)$ is linear or exponential for (arguably) too long, and wasting a day in the process, they realize that taking a third measurement might settle the debate.

c. [2 points] Based on the additional data shown to the left, circle the one best answer:

t	3	4	6
$B(t)$	16	20	25

(i) $B(t)$ is linear but not exponential

(ii) $B(t)$ is exponential but not linear

(iii) $B(t)$ is both linear and exponential

[✓ (iv)] $B(t)$ is neither linear nor exponential

Now let $S(t)$ be the amount of birdseed Charlie and Parker use, in pounds, on the t^{th} day after the start of spring.

d. [2 points] Write an equation involving B, S, and/or their inverses that represents the following statement:

Charlie and Parker use 2.3 *pounds of birdseed two days after they see* 43 *birds.*

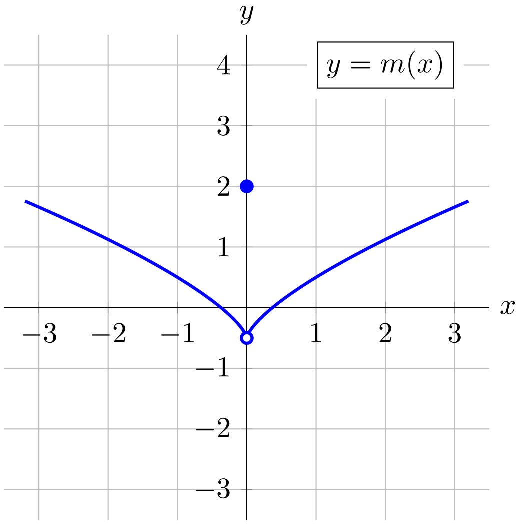
Answer: $S^{-1}(2.3) = B^{-1}(43) + 2$

6. [8 points]

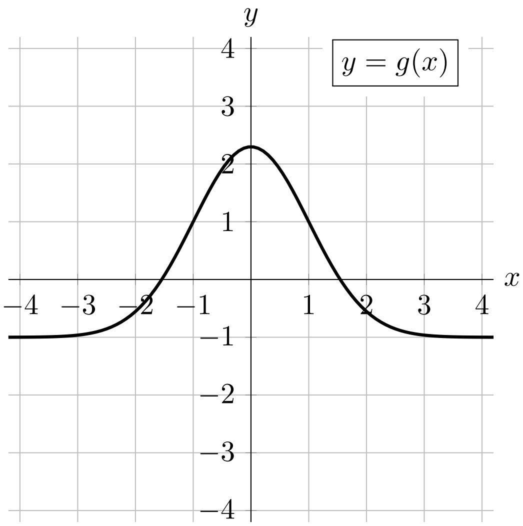
a. [4 points] Carefully draw the graph of a single function on the given axes that satisfies the given conditions, or, if no such function exists, write DNE.

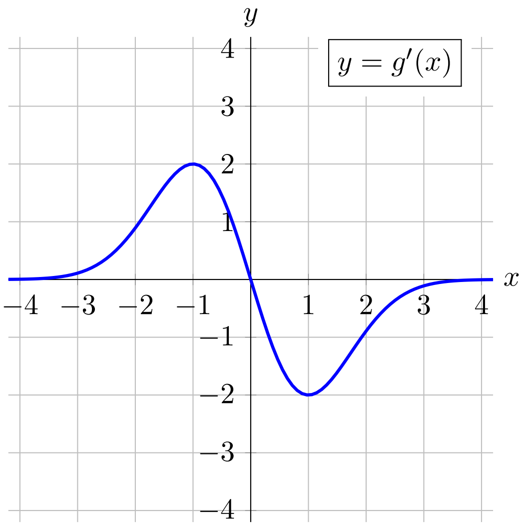
A function $m(x)$ with domain containing $(-3, 3)$ such that

- $m(x)$ is even,
- $m(x)$ is continuous and decreasing on $(-3, 0)$,
- $m(x)$ is concave down on $(0, 3)$, and
- $m(x)$ is not continuous at $x = 0$.

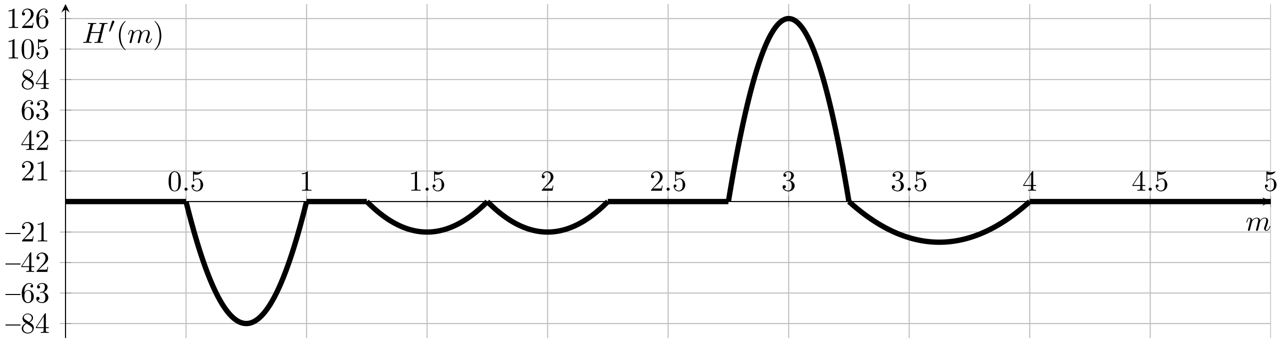


b. [4 points] A portion of the graph of the function $g(x)$ is shown below on the left. Carefully sketch the graph of $g'(x)$ for $-4 < x < 4$ on the given axes on the right.





7. [8 points] You and your friends are analyzing the vertical motion of the elevator in Mason Hall, with $H(m)$ giving the height, in feet, of the elevator m minutes after 1:00 pm. Below is a graph of $H'(m)$, the **derivative** of $H(m)$.



a. [2 points] What is the speed of the elevator at 1:02 pm? *Include units.*

Answer: 21 **feet per minute** .

b. [1 point] Find all times m for $0 < m < 5$ when the elevator is moving with its maximum speed. Give your answers as values(s) and/or interval(s) of m.

Answer: $m = 3$.

c. [1 point] For which of the following intervals is the elevator moving *downward* over the entire interval? Circle all correct choices.

[✓ $(0.5, 0.75)$] [✓ $(0.75, 1)$] $(2.75, 3)$ $(3, 3.25)$ NONE OF THESE

d. [1 point] Is the elevator's position at 1:02 pm *above*, *below*, or at the *same level* as its initial position at 1:00 pm? Circle the one correct answer.

ABOVE [✓ BELOW] SAME LEVEL

e. [1 point] Which of the following sentences best describes how the elevator is moving during the time interval $1.25 \le m \le 2$? Circle the one best choice.

[✓ (i)] *The elevator gets stuck for a moment while going down.*

(ii) *The elevator is moving up and down.*

(iii) *The elevator leaves a floor but returns to the same floor.*

(iv) *The elevator is going down without issue.*

f. [2 points] Given that the elevator's position at 1:01 pm is 28 feet away from its position at 1:00 pm, find the average velocity of the elevator over the interval $0.5 \le m \le 1$. *Include units.*

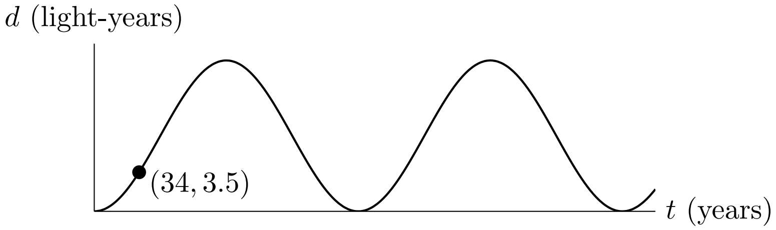
Solution: From the graph of $H'(m)$, we see that the elevator is stationary from $m = 0$ to $m = 0.5$, and then moves down from $m = 0.5$ to $m = 1$. So if at 1:01pm it ends up 28 feet away from its position at 1pm, it must have moved 28 feet downward from $m = 0.5$ to $m = 1$, which means ints average velocity over this interval is $\frac{-28}{0.5} = -56$ ft/min.

Answer: -56 **feet per minute** .

8. [8 points] Far in the future, a particular spaceship transports humans back and forth along a fixed path between the two star systems Alpha and Beta. The distance d in light-years from Alpha to the spaceship t years after its initial departure from Alpha is given by

$$d = s(t) = 6.75 - 6.75\cos\left(\tfrac{2\pi}{200}t\right).$$

A graph of $s(t)$, along with a table of a few values of both $s(t)$ and $s'(t)$, is given below.



t	0	9	12	34
$s(t)$	0	0.27	0.47	3.50
$s'(t)$	0	0.06	??	0.19

a. [1 point] How many years does it take for the spaceship to travel from Alpha to Beta?

Answer: **100** years.

b. [1 point] How many light-years apart are Alpha and Beta?

Answer: **13.5** light-years.

c. [1 point] Using the table, give the best possible estimate of $s'(12)$.

Answer: $\dfrac{0.47 - 0.27}{12 - 9}$ light-years per year.

There is a “Mystery Spot” along the spaceship’s path where gravity seems to be a little different. The ship first passes this spot 34 years after departing Alpha. (See the graph above.)

d. [3 points] A baby tortoise is born on the spaceship just as it departs Beta, and becomes the ship’s mascot, remaining aboard for the rest of its life. Assuming the tortoise lives 212 years, how many times will it get to see the Mystery Spot, and at what age(s)?

Solution: Using graph symmetry and the fact that the period of $s(t)$ is 200, the next three times after $t = 34$ that the ship passes by the Mystery Spot are $t = 166$, $t = 234$, and $t = 366$. Since the ship is at Beta at time $t = 100$, this means the ship passes by the Mystery Spot 66 years, 134 years, and 266 years after being at Beta. But the tortoise only lives 212 years, so it will only see the Mystery Spot twice, at ages 66 and 134.

Answer: **two** times, at ages 66 and 134.

e. [2 points] Now suppose the Mystery Spot wrecks the ship’s engines at time $t = 34$, so the spaceship is left to drift along forever at the speed and in direction it was going when it reached the Mystery Spot. Under this new assumption, how far away would the spaceship be from Alpha two years after it reached the Mystery Spot? *(Include units. Note that $t = 34$ appears in the graph and in the table.)*

Solution: At time $t = 34$, the spaceship is 3.5 light-years from Alpha and moving directly away from Alpha at a speed of 0.19 light-years per year. So if it keeps moving away from Alpha at this same speed, in two years it will be $2 \cdot 0.19 = 0.38$ light-years farther away, so its total distance from Alpha will be $3.5 + 0.38 = 3.88$ light-years.

Answer: $3.50 + 2 \cdot 0.19 = 3.88$ **light-years**