Math 115 — Second Midterm — November 6, 2023

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	9	
3	4	
4	8	
5	12	

Problem	Points	Score
6	6	
7	5	
8	8	
9	10	
10	8	
Total	80	

1. [10 points] Some values of the invertible, differentiable function $G(t)$ are shown in the table below, along with some values of $G'(t)$, the **derivative** of $G(t)$.

t	0	1	2	3	4	5	6
$G(t)$	0	2	5	7	8	10	11
$G'(t)$	0	5	1	2	1	3	0

For parts **a.** – **d.**, find the **exact** numerical values, or write DNE if the value does not exist. Your answers should not include the letter G, but you do not need to simply. *Show your work.*

a. [2 points] Let $P(t) = t^3 G(t)$. Find $P'(2)$.

Solution: $P'(t) = 3t^2 G(t) + t^3 G'(t)$, so

$$P'(2) = 3 \cdot 2^2 G(2) + 2^3 G'(2) = 12 \cdot 5 + 8 \cdot 1 = 68.$$

Answer: $P'(2) =$ 68

b. [2 points] Let $A(t) = \dfrac{G(3t+2)}{2t+1}$. Find $A'(1)$.

Solution: $A'(t) = \dfrac{3G'(3t+2)(2t+1) - 2G(3t+2)}{(2t+1)^2}$, so

$$A'(1) = \frac{3G'(5) \cdot 3 - 2G(5)}{3^2} = \frac{27-20}{9} = \frac{7}{9}.$$

Answer: $A'(1) =$ 7/9

c. [2 points] Let $K(t) = G^{-1}(t)$. Find $K'(2)$.

Solution: $K'(2) = \dfrac{1}{G'\big(G^{-1}(2)\big)} = \dfrac{1}{G'(1)} = \dfrac{1}{5}$.

Answer: $K'(2) =$ 1/5

d. [2 points] Let $R(t) = \ln(G(t))$. Find $R'(5)$.

Solution: $R'(t) = \frac{1}{G(t)} \cdot G'(t)$, so $R'(5) = \dfrac{1}{G(5)} \cdot G'(5) = \dfrac{3}{10}$.

Answer: $R'(5) =$ 3/10

e. [2 points] Gabby the gopher is furiously digging an underground tunnel. Suppose $G(t)$ gives the length in meters of Gabby's tunnel t hours after she started digging at 6am.

Fill in the blank with a number to give a practical interpretation of the fact that $G'(5) = 3$.

Solution: The interval from 10:55 to 11:05 is ten minutes, which is one-sixth of an hour, so we need to divide $G'(5)$ by 6.

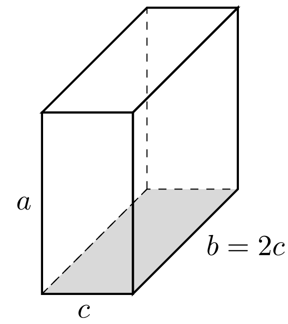
Gabby's tunnel was about 1/2 *meters longer at 11:05am than it was at 10:55am.*

2. [9 points] A company is designing a new line of suitcases with height a, length b, and width c, all in inches. The dimensions of the new suitcases need to satisfy the constraints

$$b = 2c \qquad \text{and} \qquad a + b + c = 45.$$

What are the dimensions of such a suitcase with the largest possible volume, and what is this maximum volume?

Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the volume of the suitcase.



Solution: We want to maximize volume, V, where

$$V = abc = a(2c)c = 2ac^2.$$

In order to maximize V, we must first express it in terms of a single variable. We can do this by solving for a in terms of c using the constraint equation $a + b + c = 45$:

$$a = 45 - b - c = 45 - 2c - c = 45 - 3c.$$

Substituting $a = 45 - 3c$ into our expression for V, we get

$$V = 2ac^2 = 2(45 - 3c)c^2 = 90c^2 - 6c^3.$$

Now that we have expressed V in terms of the single variable c, we can solve the problem by maximizing V over an appropriate domain. Since both c and V should be positive, we have $0 < c < 15$, so we must **maximize $V(c) = 90c^2 - 6c^3$ over the domain $(0, 15)$.**

Differentiating, we get

$$\frac{dV}{dc} = 180c - 18c^2 = 18c(10 - c).$$

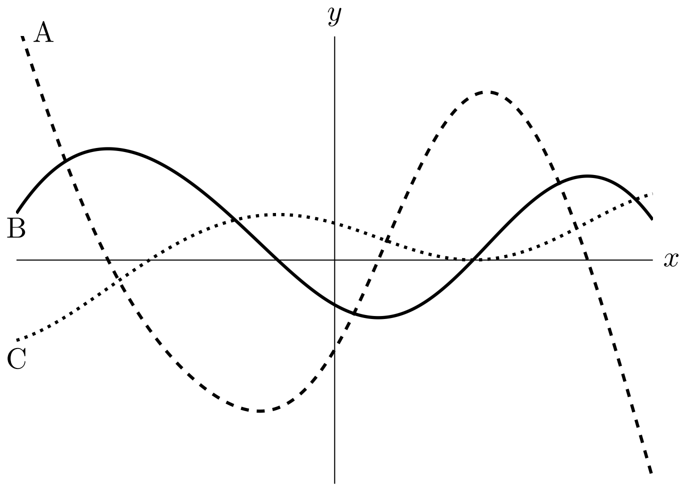
Setting $\frac{dV}{dc}$ equal to zero and solving gives us the critical points $c = 0$ and $c = 10$. Since $90c^2 - 6c^3$ is differentiable everywhere, there are no other critical points.

Now we test V at $c = 0, 10, 15$. Since $V(0) = V(15) = 0$ and $V(10) = 2(45 - 3 \cdot 10) \cdot 10^2 = 3000$, we see that the maximum of $V(c)$ on $(0, 15)$ occurs at $c = 10$, with a value of $V(10) = 3000$. Solving for a and b using the constraint equations gives $b = 20$ and $a = 15$.

Answer: The volume is maximized when $a =$ 15 in., $b =$ 20 in., and $c =$ 10 in.,

and the maximum volume is 3000 cubic inches.

3. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $f(x)$: C

$f'(x)$: B

$f''(x)$: A

4. [8 points] Suppose $f(x)$ and $g(x)$ are functions that have exactly the same four critical points, namely at $x = 1$, $x = 3$, $x = 5$, and $x = 7$. Note that f and g have **no other** critical points beyond these four. Assume the first and second derivatives of $f(x)$ and $g(x)$ exist everywhere.

The table below shows some values of $f'(x)$ and $g''(x)$ at certain inputs. Note that the table gives values of the **first derivative** of $f(x)$ and the **second derivative** of $g(x)$.

x	0	1	2	3	4	5	6	7	8
$f'(x)$	3	0	-1	0	1	0	2	0	?
$g''(x)$	?	0	-1	-4	?	0	?	2	1

a. [4 points] Use the table to classify each critical point of f as a local minimum, maximum, or neither of f. Circle your answer. If there is not enough information to decide, circle NEI.

i. $x = 1$ is a LOCAL MIN of f **LOCAL MAX of f** NEITHER NEI

ii. $x = 3$ is a **LOCAL MIN of f** LOCAL MAX of f NEITHER NEI

iii. $x = 5$ is a LOCAL MIN of f LOCAL MAX of f **NEITHER** NEI

iv. $x = 7$ is a LOCAL MIN of f LOCAL MAX of f NEITHER **NEI**

b. [4 points] Use the table to classify each critical point of g as a local minimum, maximum, or neither of g. Circle your answer. If there is not enough information to decide, circle NEI.

i. $x = 1$ is a LOCAL MIN of g LOCAL MAX of g NEITHER **NEI**

ii. $x = 3$ is a LOCAL MIN of g **LOCAL MAX of g** NEITHER NEI

iii. $x = 5$ is a LOCAL MIN of g LOCAL MAX of g **NEITHER** NEI

iv. $x = 7$ is a **LOCAL MIN of g** LOCAL MAX of g NEITHER NEI

Solution: Part **a.** follows from the First Derivative Test, and most of **b.** from the Second Derivative Test. For **b.**(iii.), note that g must be decreasing on both $(3, 5)$ and $(5, 7)$ since $x = 5$ is the only critical point of g on $(3, 7)$ and we have $g''(3) < 0$ but $g''(7) > 0$.

5. [12 points] A continuous function $h(x)$, its derivative $h'(x)$, and its second derivative $h''(x)$ are given by

$$h(x) = \frac{x}{x^2+1}, \qquad h'(x) = \frac{1-x^2}{(x^2+1)^2}, \qquad \text{and} \qquad h''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

Note that the critical points of $h(x)$ are ± 1, and the critical points of $h'(x)$ are 0 and $\pm\sqrt{3}$.

For each part below, you must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

a. [4 points] Find the x-coordinates of all global minima and global maxima of $h(x)$ **on the interval** $[0, 2]$. If there are none of a particular type, write NONE.

Solution: We need to evaluate h at the endpoints of $[0, 2]$ and the critical points of h in $[0, 2]$, and find the greatest and least of these values. The critical points of h are ± 1, so $x = 1$ is the only critical point of h in $[0, 2]$. Since

$$h(0) = 0, \qquad h(1) = \frac{1}{2}, \qquad \text{and} \quad h(2) = \frac{2}{5},$$

on the interval $[0, 2]$ the function h has a global min of 0 at $x = 0$, and a global max of $\frac{1}{2}$ at $x = 1$.

Answer: Global min(s) at $x =$ <u>0</u> and Global max(es) at $x =$ <u>1</u>

b. [4 points] Find the x-coordinates of all global minima and global maxima of $h(x)$ on the interval $(-\infty, \infty)$. If there are none of a particular type, write NONE.

Solution: We need to evaluate h at its critical points $x = \pm 1$, and also find its end behavior as $x \to \pm\infty$. We have $h(1) = \frac{1}{2}$ and $h(-1) = -\frac{1}{2}$, while

$$\lim_{x\to-\infty} h(x) = 0 = \lim_{x\to\infty} h(x).$$

This means that h has a global min of $-\frac{1}{2}$ at $x = -1$, and a global max of $\frac{1}{2}$ at $x = 1$.

Answer: Global min(s) at $x =$ <u>-1</u> and Global max(es) at $x =$ <u>1</u>

c. [4 points] Find the x-coordinates of all inflection points of $h(x)$ on the interval $(-\infty, \infty)$.

Solution: Since h is continuous, its inflection points will occur where its concavity changes, that is, where its second derivative changes sign. So we make a sign chart for $h''(x)$, which is zero at 0 and $\pm\sqrt{3}$:

$h''(x)$: $\frac{-\cdot+}{+} = -$ | $-\sqrt{3}$ | $\frac{-\cdot-}{+} = +$ | 0 | $\frac{+\cdot-}{+} = -$ | $\sqrt{3}$ | $\frac{+\cdot+}{+} = +$

$\frown$ $-\sqrt{3}$ $\smile$ 0 $\frown$ $\sqrt{3}$ $\smile$

Because h'' changes sign at each of the points $x = 0, \pm\sqrt{3}$, all three of these points are inflection points of h.

Answer: Inflection point(s) at $x =$ <u>$-\sqrt{3}$, 0, $\sqrt{3}$</u>

6. [6 points] Let $\mathcal{C}$ be the curve implicitly defined by the equation $xy = y^2 + 2x$. Note that

$$\frac{dy}{dx} = \frac{2-y}{x-2y}.$$

a. [3 points] Find the coordinates of all points on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is horizontal. If no such points exist, write DNE and show work to justify your answer.

Solution: The curve $\mathcal{C}$ has a horizontal tangent line at points where $\frac{dy}{dx} = 0$. From the given equation $\frac{dy}{dx} = \frac{2-y}{x-2y}$, we see that this happens when $y = 2$ and $x \neq 4$. Substituting $y = 2$ into the equation that defines $\mathcal{C}$, we get

$$2x = 2^2 + 2x,$$

which has no solutions. It follows that there are *no* points on the curve $\mathcal{C}$ where $\mathcal{C}$ has a horizontal tangent line.

Answer: **DNE**

b. [3 points] Find the coordinates of all points on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is vertical. If no such points exist, write DNE and show work to justify your answer.

Solution: The curve $\mathcal{C}$ has a vertical tangent line at points where $\frac{dx}{dy} = 0$. Since $\frac{dx}{dy} = \frac{x-2y}{2-y}$, this happens when $x = 2y$ and $y \neq 2$. Substituting $x = 2y$ into the equation that defines $\mathcal{C}$ gives

$$2y^2 = y^2 + 4y, \qquad \text{which is equivalent to} \quad y(y-4) = 0.$$

This has solutions $y = 0$ and $y = 4$. Plugging these values for y into $xy = y^2 + 2x$, we see that $x = 0$ when $y = 0$, and $x = 8$ when $y = 4$.

Answer: $(0,0)$ **and** $(8,4)$

7. [5 points] The equation $\sin(x^3) + x^2 y = 1 + y^2$ defines y implicitly as a function of x.

Find a formula for $\frac{dy}{dx}$ in terms of x and y. Show every step of your work.

Solution: Implicitly differentiating this equation with respect to x gives

$$\cos(x^3)\cdot 3x^2 + 2xy + x^2\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

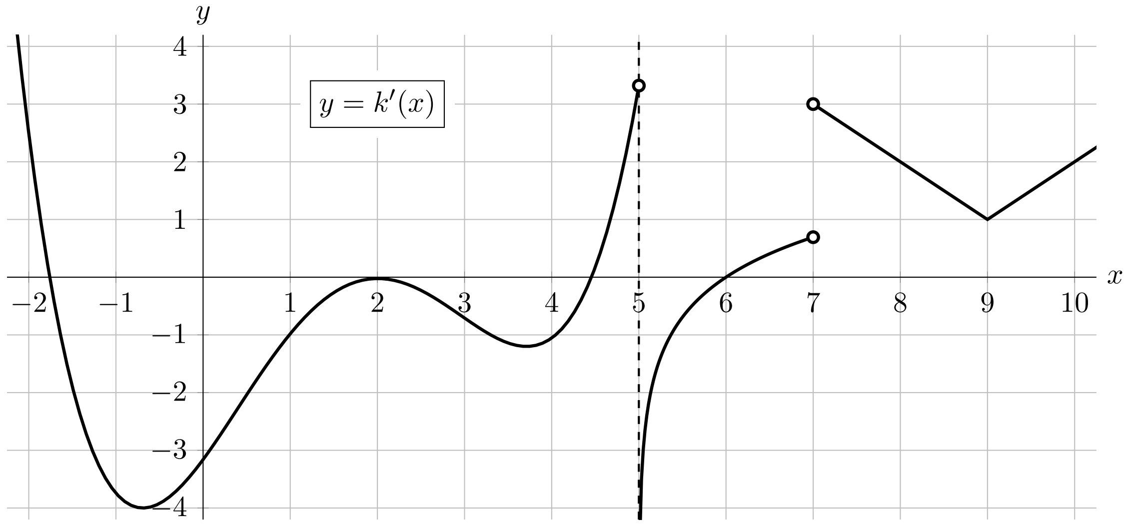
Collecting the terms involving $\frac{dy}{dx}$ on one side, we get

$$\cos(x^3)\cdot 3x^2 + 2xy = 2y\frac{dy}{dx} - x^2\frac{dy}{dx} = (2y - x^2)\frac{dy}{dx}.$$

Finally, after dividing both sides by $2y - x^2$ we get $\frac{dy}{dx} = \frac{3x^2\cos x^3 + 2xy}{2y - x^2}$.

Answer: $\frac{dy}{dx} = \frac{dy}{dx} = \frac{3x^2\cos x^3 + 2xy}{2y - x^2}$

8. [8 points] Suppose $k(x)$ is a continuous function, defined for all real numbers. A portion of the graph of $k'(x)$, the **derivative** of $k(x)$, is given below. Note that $k'(x)$ has a vertical asymptote at $x = 5$ and a sharp corner at $x = 9$.



a. [2 points] Circle the least value that is listed below.

$k(-1)$ $k(0)$ $k(1)$ $k(2)$ $\boxed{k(3)}$

b. [2 points] Circle the least value that is listed below.

$\boxed{k''(-2)}$ $k''(-1)$ $k''(0)$ $k''(1)$ $k''(2)$

c. [2 points] Circle all points listed below that are inflection points of $k(x)$.

$x = \frac{1}{2}$ $\boxed{x = 2}$ $x = 3$ $x = 6$ $\boxed{x = 9}$ NONE OF THESE

d. [1 point] On which of the following intervals does $k'(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers.

$\boxed{[-1, 3]}$ $[3, 5]$ $[6, 8]$ $\boxed{[8, 9]}$ $[8, 10]$ NONE OF THESE

e. [1 point] On which of the following intervals does $k(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers.

$\boxed{[-1, 3]}$ $\boxed{[3, 5]}$ $[6, 8]$ $\boxed{[8, 9]}$ $\boxed{[8, 10]}$ NONE OF THESE

9. [10 points] The continuous function $w(x)$ is defined piecewise for all real numbers by the rule

$$w(x) = \begin{cases} -x^2+3x+1 & x<-1 \\ 3x^{1/3} & -1\le x\le 1 \\ -x^2+3x+1 & x>1. \end{cases}$$

a. [5 points] Find the x-coordinates of all critical points of $w(x)$. If there are none, write NONE. Show your work.

Solution: We must find all x for which $w'(x)$ is either zero or does not exist. We have

$$w'(x) = \begin{cases} 3-2x & x<1; \\ x^{-2/3} & -1<x<1; \\ 3-2x & x>1; \\ \text{DNE} & x=-1; \\ 1 & x=1. \end{cases}$$

Note that $w'(-1)$ DNE since $3-2(-1)\neq(-1)^{-2/3}$, while $w'(1)=1$ since $3-2(1)=1=(1)^{-2/3}$. Now, $3-2x=0$ when $x=\frac{3}{2}$, and $x^{-2/3}$ is never zero but is undefined when $x=0$. Thus the critical points of $w(x)$ are:

$$x=-1,\ x=0,\ \text{and } x=3/2.$$

Answer: Critical point(s) at $x =$ **-1, 0, and $3/2$**

b. [3 points] Let $L(x)$ be the linear approximation of the function $w(x)$ at the point $x=\frac{1}{2}$. Find a formula for $L(x)$. Your answer should not include the letter w, but you do not need to simplify.

Solution: The linear approximation of $w(x)$ at $x=\frac{1}{2}$ is

$$\begin{aligned} L(x) &= w\left(\frac{1}{2}\right)+w'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right) \\ &= 3\left(\frac{1}{2}\right)^{1/3}+\left(\frac{1}{2}\right)^{-2/3}\left(x-\frac{1}{2}\right) = \frac{3}{\sqrt[3]{2}}+\sqrt[3]{4}\left(x-\frac{1}{2}\right). \end{aligned}$$

Answer: $L(x) = 3\left(\frac{1}{2}\right)^{1/3}+\left(\frac{1}{2}\right)^{-2/3}\left(x-\frac{1}{2}\right)$

c. [2 points] Does $L(x)$ give an overestimate or underestimate for $w(x)$ near $x=\frac{1}{2}$? Circle your answer below, and show work to justify your answer.

UNDERESTIMATE OVERESTIMATE

Solution: $L(x)$ gives an overestimate for $w(x)$ near $x=\frac{1}{2}$ if $w(x)$ is concave down near $\frac{1}{2}$, and an underestimate if $w(x)$ is concave up near $\frac{1}{2}$. We can determine the concavity of $w(x)$ near $\frac{1}{2}$ by looking at the sign of $w''(x)$. Near $x=\frac{1}{2}$, we have

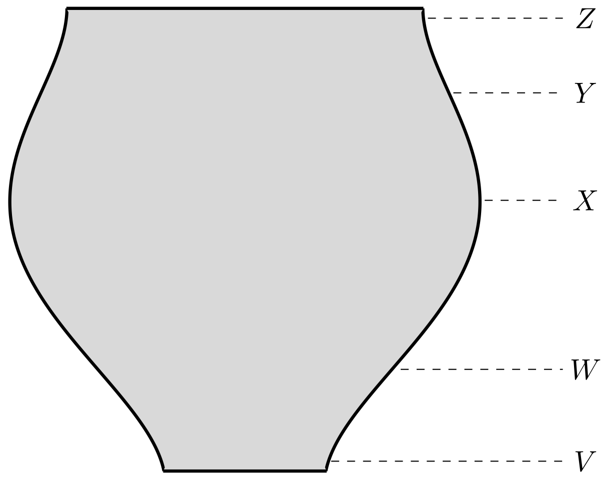
$$w'(x)=x^{-2/3}, \quad \text{so} \quad w''(x)=-\left(\frac{2}{3}\right)x^{-4/3} \quad \text{and} \quad w''\left(\frac{1}{2}\right)=-\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{-4/3}<0.$$

Thus $w(x)$ is **concave down** near $x=\frac{1}{2}$, so $L(x)$ gives an **overestimate**.

10. [8 points] Water is pouring at a constant positive rate into a circular planter of height 40 inches, whose profile from the side is displayed below. For $0 \leq t \leq 10$, let $D(t)$ be the depth in inches of the water in the planter t minutes after water first starts pouring into the planter.

Assume the first and second derivatives of $D(t)$ exist and are continuous on the interval $(0, 10)$. We know that it takes exactly ten minutes for the water to fill the planter completely, so $D(0) = 0$ and $D(10) = 40$.

Let v, w, x, y, z be the times, in minutes, that it takes the water level in the planter to reach the heights V, W, X, Y, and Z, respectively, that are shown in the figure. So, for instance, $Y = D(y)$. Note that X is the height at which the planter is the widest, and heights W and Y correspond to inflection points in the curve that gives the profile of the planter.



a. [2 points] Determine whether each statement below is true or false. Indicate your answer by clearly writing TRUE or FALSE on the blank before each statement.

(i) ____TRUE____ The function $D(t)$ is increasing on the interval $[0, 10]$.

(ii) ____TRUE____ The function $D(t)$ is invertible on the interval $[0, 10]$.

b. [1 point] How does $D(5)$ compare with 20? Circle the correct statement below.

$D(5) < 20$ $D(5) = 20$ **[$D(5) > 20$]**

c. [1 point] Circle all points below at which the derivative $D'(t)$ attains a global **maximum** on the interval $[v, z]$.

[v] w x y z NONE OF THESE

d. [1 point] Circle all points below at which the derivative $D'(t)$ attains a global **minimum** on the interval $[v, z]$.

v w **[x]** y z NONE OF THESE

e. [1 point] Circle all intervals below on which the derivative $D'(t)$ is **increasing**.

(v, w) (w, x) **[(x, y)]** **[(y, z)]** NONE OF THESE

f. [1 point] Circle all intervals below on which the function $D(t)$ is **concave up**.

(v, w) (w, x) **[(x, y)]** **[(y, z)]** NONE OF THESE

g. [1 point] Circle all **inflection points** of the function $D(t)$ on the interval $(0, 10)$.

v w **[x]** y z NONE OF THESE