Math 115 — Final Exam — December 8, 2023

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score	Problem	Points	Score
1	14		6	15	
2	12		7	9	
3	10		8	11	
4	6		9	10	
5	5		10	8	
			Total	100	

1. [14 points] Given below is a table of some values of the **even** function $k(x)$, along with its derivative $k'(x)$. Assume the functions $k(x)$, $k'(x)$, and $k''(x)$ are continuous on $(-\infty, \infty)$, and that $k(x)$ is decreasing on $(0, \infty)$. Your final answers in this problem should not include the letter k.

x	0	1	2	3	4	5
$k(x)$	12	8	7	2	0	-3
$k'(x)$	0	-3	-4	-2	-1	-5

In parts **a.–c.**, find the numerical value **exactly**, or write NEI if there is not enough information provided to do so. *Show your work. Limited partial credit may be awarded for work shown.*

a. [2 points] Find $\displaystyle\int_1^3 \left(2k'(x) + e^x\right) dx$.

Solution:

$$\int_1^3 \left(2k'(x)+e^x\right) dx \;=\; 2\int_1^3 k'(x)\,dx+\int_1^3 e^x\,dx \;=\; 2\big(k(3)-k(1)\big)+\left(e^3-e^1\right) \;=\; 2(2-8)+e^3-e.$$

Answer: $-12 + e^3 - e$

b. [2 points] Find the average value of $k''(x)$ on the interval $[1, 4]$.

Solution: The average value of $k''(x)$ on $[1, 4]$ is

$$\frac{1}{4-1}\int_1^4 k''(x)\,dx \;=\; \frac{1}{3}\big(k'(4)-k'(1)\big) \;=\; \frac{-1-(-3)}{3} \;=\; \frac{2}{3}.$$

Answer: $2/3$

c. [2 points] Find $\displaystyle\lim_{h\to 0} k(-1) + \lim_{h\to 0} \frac{k(3+h)-k(3)}{h}$.

Solution: $\displaystyle\lim_{h\to 0} k(-1) = k(-1) = k(1) = 8$ and $\displaystyle\lim_{h\to 0}\frac{k(3+h)-k(3)}{h} = k'(3) = -2$, so

$$\lim_{h\to 0} k(-1) + \lim_{h\to 0}\frac{k(3+h)-k(3)}{h} \;=\; 8+(-2) \;=\; 6.$$

Answer: 6

d. [2 points] Use the table to estimate $k''(4.5)$.

Solution: $\displaystyle k''(4.5) \;\approx\; \frac{k'(5)-k'(4)}{5-4} \;=\; \frac{-5-(-1)}{5-4} \;=\; -4.$

Answer: -4

This problem continues on the next page.

This problem continues from the previous page. The table of some values of the even function $k(x)$ and its derivative $k'(x)$ is displayed again for convenience. Recall that $k(x)$, $k'(x)$, and $k''(x)$ are continuous on $(-\infty, \infty)$, and $k(x)$ is decreasing on $(0, \infty)$. Your final answers in this problem should not include the letter k.

x	0	1	2	3	4	5
$k(x)$	12	8	7	2	0	-3
$k'(x)$	0	-3	-4	-2	-1	-5

e. [2 points] Find the linear approximation $L(x)$ of the function $j(x) = 3k(2x) + 1$ at the point $x = 2$.

Solution: We have $j(2) = 3k(4) + 1 = 3 \cdot 0 + 1 = 1$ and $j'(2) = 6k'(4) = -6$, so

$$L(x) = j(2) + j'(2)(x-2) = 1 - 6(x-2) = -6x + 13.$$

Answer: $L(x) =$ $1 - 6(x-2)$

f. [2 points] Use a right-hand Riemann sum with two equal subdivisions to estimate $\int_1^5 k(x)\,dx$. Write out all the terms in your sum, which you do not need to simplify.

Solution:

$$2 \cdot k(3) + 2 \cdot k(5) = 2 \cdot 2 + 2 \cdot (-3) = -2.$$

g. [2 points] Does the sum described in part **f.** *overestimate*, *underestimate*, or *equal* the value of

$$\int_1^5 k(x)\,dx?$$

Circle your answer and provide a brief explanation. If there is not enough information to decide, circle NEI.

Circle one: OVERESTIMATE $\boxed{\text{UNDERESTIMATE}}$ EQUAL NEI

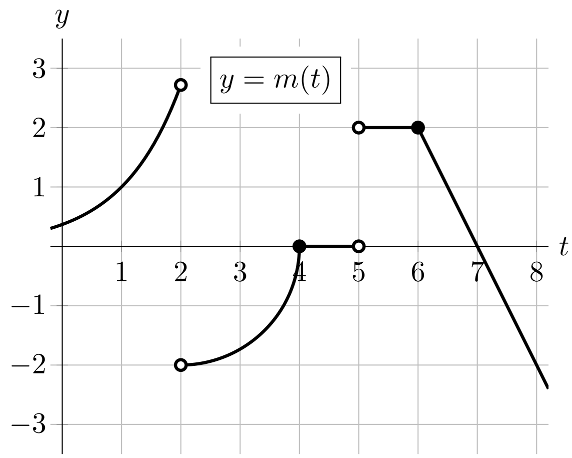
Explanation:

Solution: $k(x)$ is decreasing, and a right-hand Riemann sum of a decreasing function always underestimates the corresponding integral.

2. [12 points]

A portion of the graph of the function $m(t)$ is shown to the right. Note the following facts about $m(t)$:

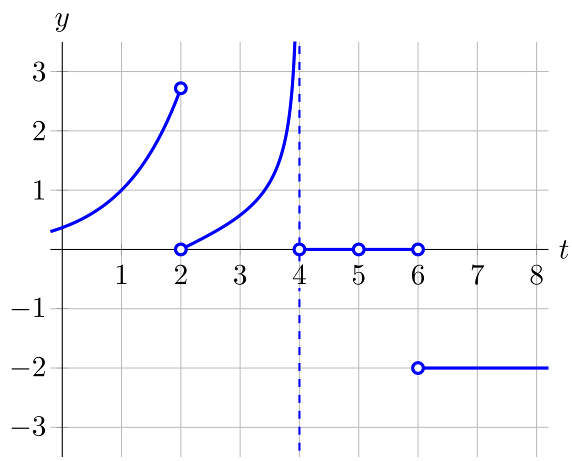
- on the interval $t < 2$, we have $m(t) = e^{t-1}$;
- on the interval $2 < t < 4$, the graph of $m(t)$ is a quarter of a circle;
- $m(t)$ is piecewise linear on the interval $4 < t < 6$;
- $m(t)$ is linear on the interval $t > 6$.



a. [5 points]

On the axes to the right, sketch a detailed graph of $m'(t)$, the <u>derivative</u> of $m(t)$, for $0 \le t \le 8$. Make sure the following are clear from your graph:

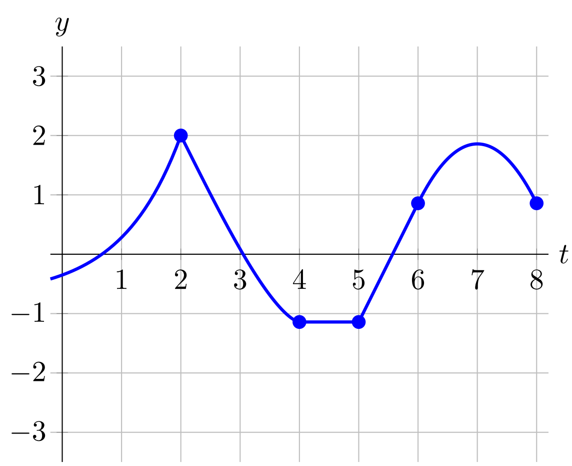
- where $m'(t)$ is undefined;
- any vertical asymptotes of $m'(t)$;
- where $m'(t)$ is zero, positive, or negative;
- where $m'(t)$ is increasing, decreasing, or constant;
- where $m'(t)$ is linear (with correct slope).



b. [7 points]

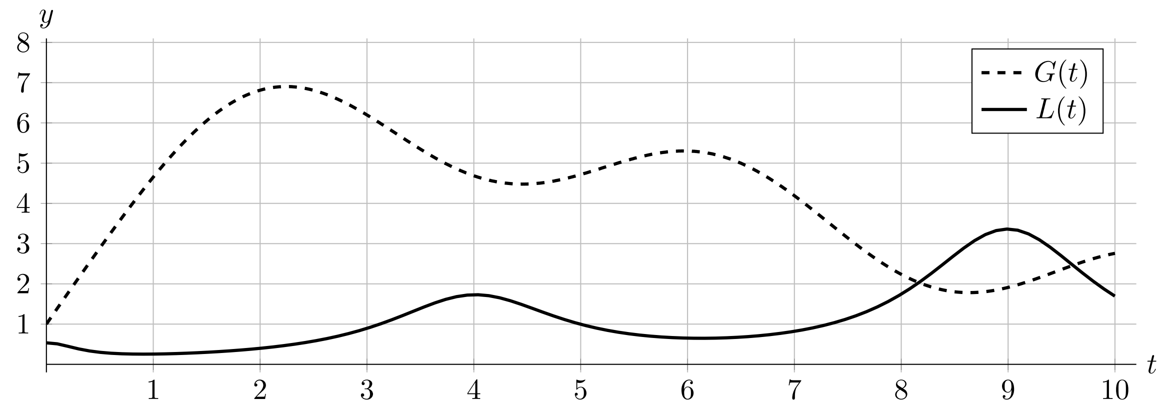
Let $M(t)$ be a continuous <u>antiderivative</u> of $m(t)$ satisfying $M(2) = 2$. On the axes to the right, sketch a detailed graph of $M(t)$ for $0 \le t \le 8$. Note that $\pi \approx 3.14$, $e \approx 2.72$ and $e^{-1} \approx 0.37$. Make sure the following are clear from your graph:

- where $M(t)$ is and is not differentiable;
- the approximate values of $M(t)$ at $t = 0$, 3, 4, 5, 6, 7, and 8;
- where $M(t)$ is increasing, decreasing, and constant;
- the concavity and any inflection points of $M(t)$.



3. [10 points] Television ratings for the popular sport of pigglystick have been fluctuating wildly this fall, ever since mega popstar Becky Twist started dating one of the sport's premier players. Becky's fans ("Twisties") have flocked to the sport in droves, producing a backlash among some longtime fans who have started tuning out.

Let $G(t)$ model the rate at which pigglystick has attracted new fans, and $L(t)$ the rate at which the sport has lost fans, in millions of fans per week, t weeks after the two celebrities started dating on October 1st. Graphs of the continuous functions $G(t)$ and $L(t)$ are shown below.



a. [2 points] Write an expression for the total number of new fans pigglystick attracted, in millions, over the first three weeks of October.

Answer: $\int_0^3 G(t)\,dt$ million fans.

b. [2 points] Given that there were an estimated 200 million fans of pigglystick on December 3rd, which was exactly nine weeks after October 1st, write an expression for the number of fans pigglystick had on October 1st, in millions.

Answer: $200 - \int_0^9 \big(G(t) - L(t)\big)\,dt$ million fans.

c. [2 points] Estimate the rate, in millions of fans per week, at which the number of fans of pigglystick was changing on November 5th, which is five weeks after October 1st.

Answer: ≈ 3.7 .

d. [2 points] Estimate the time, in number of weeks after October 1st, when pigglystick had the *most* fans.

Answer: ≈ 8.2

e. [2 points] Write an expression for the average rate of change in the number of fans of pigglystick over the ten weeks following October 1st, in millions of fans per week.

Answer: $\frac{1}{10}\int_0^{10} \big(G(t) - L(t)\big)\,dt$

4. [6 points] The expressions below define five different functions of x, labeled i. – v., which are used for answers in parts **a**. and **b**.

i. $\dfrac{x^2+1}{x-3}$ ii. $\dfrac{x^2-1}{x^3-8}$ iii. $\dfrac{x}{e^x}$ iv. $\dfrac{e^x}{x}$ v. $\dfrac{(x-2)^{1/3}}{x}$

a. [2 points] Which of the above functions have a limit of 0 as $x \to \infty$? *Circle all correct answers.*

i. $\boxed{\text{ii.}}$ $\boxed{\text{iii.}}$ iv. $\boxed{\text{v.}}$ NONE OF THESE

b. [2 points] Which of the above functions satisfy the hypotheses of the Mean Value Theorem on the interval $[1,3]$? *Circle all correct answers.*

i. ii. $\boxed{\text{iii.}}$ $\boxed{\text{iv.}}$ v. NONE OF THESE

c. [2 points] Circle the *one correct equality below* to complete the statement of the Mean Value Theorem applied to the function $f(x) = e^x/x$ on the interval $[1,3]$:
"There is a point c in the interval $(1,3)$ such that …"

$\dfrac{xe^x - e^x}{x^2} = \dfrac{\frac{1}{3}e^c - e}{2}$ $\dfrac{ce^c - e^c}{c^2} = \dfrac{e^3}{9}$ $\dfrac{e^c}{c} = \dfrac{\frac{1}{3}e^3 - e}{3-1}$ $\boxed{\dfrac{e^c(c-1)}{c^2} = \dfrac{e^3}{6} - \dfrac{e}{2}}$

5. [5 points] Find the global maximum and minimum output values of the function

$$y \;=\; w(x) \;=\; 3 - 2x^2 + \frac{x^4}{2}$$

on the interval $[-1,2]$, and list *all* x-values at which the global maximum and minimum occur. *Show all your work, and use calculus to justify your answers, which should be numerical.*

Solution: First we find the critical points of $w(x)$. Since $w(x)$ is differentiable everywhere, in order to find the critical points we just have to solve $w'(x) = 0$. We have

$$w'(x) \;=\; -4x + 2x^3 \;=\; 2x(x^2-2) \;=\; 2x(x-\sqrt{2})(x+\sqrt{2}).$$

Thus the critical points of $w(x)$ are $x = 0, \pm\sqrt{2}$. Now we evaluate $w(x)$ at the endpoints of $[-1,2]$, and at its critical points that belong to $[-1,2]$. We have

$$w(-1) = \frac{3}{2}, \qquad w(0) = 3, \qquad w(\sqrt{2}) = 1, \qquad w(2) = 3.$$

The greatest of these values is the maximum of $w(x)$ on $[-1,2]$, and the least is the minimum. So, on the interval $[-1,2]$, $w(x)$ has a max of $y = 3$ which occurs at $x = 0$ and $x = 2$, and a min of $y = 1$ which occurs at $x = \sqrt{2}$.

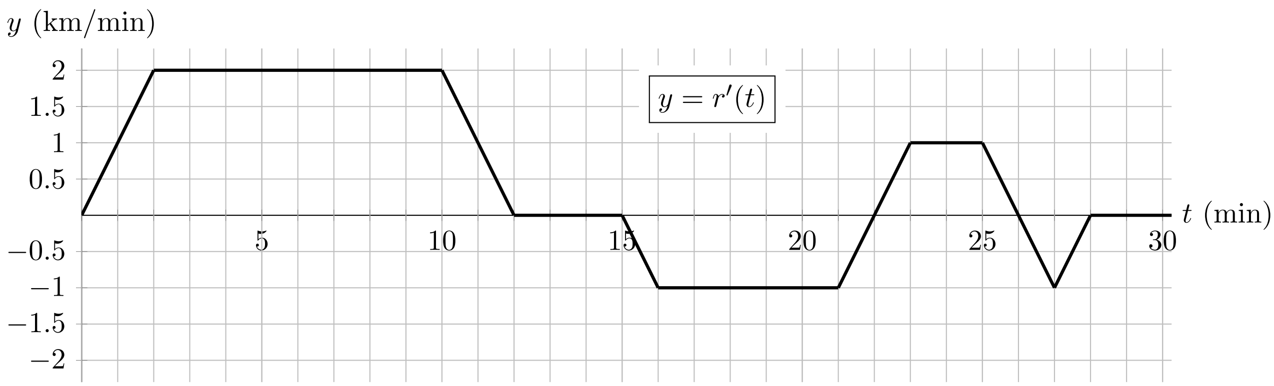
Answer: On the interval $[-1,2]$, the function $w(x)$ has a …

global maximum value of $y =$ __3__, occuring at the point(s) $x =$ __0, 2__,

and a global minimum value of $y =$ __1__, occuring at the point(s) $x =$ __$\sqrt{2}$__.

6. [15 points] Caroline, an amateur astronomer, is driving at night along a straight road through the desert between the small towns of Tycho and Brahe, trying to find the darkest spot between them in order to obtain the best viewing conditions. Brahe is 37 kilometers (km) east of Tycho.

Let $r(t)$ be Caroline's position along the road, in kilometers east of Tycho, t minutes after she departs Tycho at 9pm. At 9:28pm, she decides she is close enough to the darkest spot, and she parks her car and sets up her telescope. Pictured below is a graph of $r'(t)$, the **derivative** of $r(t)$.



a. [1 point] How many times did Caroline turn around and retrace part of her path, before eventually coming to a stop?

Answer: She turned around **3** times.

b. [2 points] Given that Tycho and Brahe are 37 km apart, what was the closest Caroline came to Brahe? That is, what was her *minimum distance from Brahe*, over the course of her drive?

Solution: Caroline drove $\int_0^{12} r'(t)\,dt = 20$ km east, then $-\int_{15}^{22} r'(t)\,dt = 6$ km west, followed by $\int_{22}^{26} r'(t)\,dt = 3$ km east and $-\int_{26}^{28} r'(t)\,dt = 1$ km west. It follows that she was furthest east (i.e., closest to Brahe) when she turned around for the first time, 20 km east of Tycho, which is 17 km west of Brahe.

Answer: The closest Caroline came to Brahe was **17** kilometers.

c. [2 points] The road Caroline is driving on crosses railroad tracks exactly 10 km east of Tycho. At what time did Caroline first cross these railroad tracks?

Solution: We must find the least $b > 0$ such that $\int_0^{b} r'(t)\,dt = 10$. By counting squares (or adding up areas of triangles and rectangles), we see that $b = 6$. So Caroline crossed the tracks 6 minutes after 9pm.

Answer: She first crossed the tracks at **9:06 pm** .

d. [2 points] Write an expression *involving one or more integrals* for the total distance in kilometers that Caroline traveled while searching for the darkest spot.

Answer: $\int_0^{28} |r'(t)|dt$, **or** $\int_0^{12} r'(t)dt - \int_{15}^{22} r'(t)dt + \int_{22}^{26} r'(t)dt - \int_{26}^{28} r'(t)dt$

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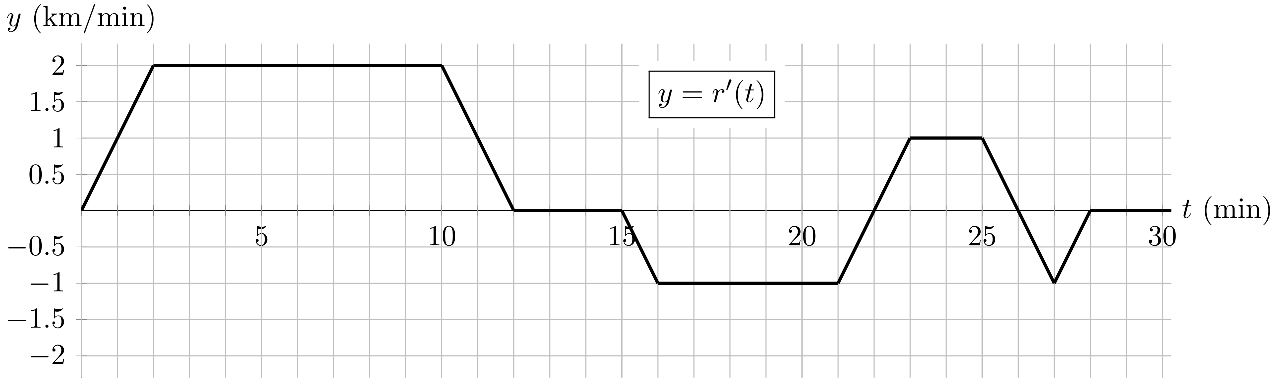
This problem continues from the previous page. The graph of $r'(t)$ is displayed again for convenience. Recall that Brahe is 37 kilometers east of Tycho.



After a while, Caroline realizes that she could have found the darkest spot between Tycho and Brahe *exactly* by solving an optimization problem, since the apparent brightness of a light source is directly proportional to the light's brightness and inversely proportional to the square of the observer's distance from the light source.

e. [3 points] Assuming that Brahe is twice as bright as Tycho, and that no other towns or light sources are near enough to be significant, find a function $f(x)$ that models, in appropriate units, the apparent brightness of the two towns at a point x km east of Tycho.

Note: you do not need to minimize $f(x)$, or give units.

Solution: Since Brahe is twice as bright as Tycho, there is some positive constant c such that, in appropriate units, the apparent brightness of Tycho x km east of Tycho is $\frac{c}{x^2}$ and the apparent brightness of Brahe x km east of Tycho is $\frac{2c}{(37-x)^2}$, so the total apparent brightness of the two towns at that point is

$$f(x) = \frac{c}{x^2} + \frac{2c}{(37-x)^2}.$$

This is a correct answer, although the proportionality constant can be eliminated by changing units, and does not affect the minimum anyway, so one could also answer more simply:

Answer: $f(x) = \dfrac{1}{x^2} + \dfrac{2}{(37-x)^2}$

f. [1 point] In order to find the darkest point, over what domain should the function $f(x)$ be minimized?

Answer: $[0, 37]$

g. [4 points] Caroline uses calculus to minimize $f(x)$, and finds the darkest point between the towns to be exactly d km east of Tycho. Happily, this point turns out to be just 0.4 km east of where she actually parked. Letting $g(t) = f(r(t))$, determine the *signs* of the quantities below by clearly writing $<$, $=$, or $>$ in the given boxes.

i. $f'(d)$ $\boxed{=}$ 0 ii. $g'(3)$ $\boxed{<}$ 0 iii. $g'(11)$ $\boxed{>}$ 0

iv. $g'(14)$ $\boxed{=}$ 0 v. $g'(16)$ $\boxed{<}$ 0 vi. $g'(22)$ $\boxed{=}$ 0

Solution: Note that $f'(d) = 0$ because d is a local extremum of f. Furthermore, $g'(t) = 0$ whenever Caroline's velocity, $r'(t)$, is zero, and $g'(t)$ is negative whenever Caroline is moving *towards* the darkest spot, and positive whenever she is moving *away* from it.

7. [9 points] In an unexpected twist, Carson Soltonni also runs a business selling vacuum cleaners out of his house. The cost in hundreds of dollars for him to produce q hundred vacuum cleaners is

$$C(q) \;=\; \frac{q^3}{3} - 5q^2 + 59q + 5.$$

Carson sells his vacuum cleaners for 50 dollars each, and he is trying to determine how many to sell in order to maximize profit. Some values of $C(q)$, rounded to the nearest integer, are given below.

q	1	2	3	4	5	6	7	8	9
$C(q)$	59	106	146	182	217	251	287	328	374

a. [1 point] What is the fixed cost of Carson's business?

Answer: 5 hundred dollars.

b. [3 points] Find the marginal revenue function $MR(q)$ and marginal cost function $MC(q)$ of Carson's business, in hundreds of dollars per hundred vacuum cleaners.

Solution: The marginal revenue and marginal cost functions are the derivatives of the revenue and cost functions, respectively. Since Carson sells vacuum cleaners for 50 dollars each, his marginal revenue is \$50 per vacuum cleaner, or, equivalently, 50 hundred dollars per hundred vacuum cleaners. And the marginal cost function will be $MC(q) = C'(q) = q^2 - 10q + 59$.

Answer: $MR(q) =$ 50 and $MC(q) =$ $q^2 - 10q + 59$

c. [3 points] How many vacuum cleaners should Carson produce and sell to maximize profit? *Show your work and use calculus.* You do not need to fully justify your answer, but partial credit may be awarded for work shown.

Solution: Carson's profit, $\pi(q)$, is equal to revenue minus cost, that is, $\pi(q) = R(q) - C(q)$. We want to maximize $\pi(q)$ over the interval $[0, \infty)$. Since $\pi(q)$ is differentiable everywhere, its critical points will occur when $\pi'(q) = 0$, that is, when $MR(q) = MC(q)$. We have

$$\pi'(q) \;=\; MR(q) - MC(q) \;=\; 50 - (q^2 - 10q + 59) \;=\; -(q^2 - 10q + 9) \;=\; -(q-1)(q-9),$$

so the critical points of $\pi(q)$ occur at $q = 1$ and $q = 9$. Checking the endpoints, we have

$$\pi(0) = -5 \qquad \text{and} \qquad \lim_{q\to\infty} \pi(q) = \lim_{q\to\infty} (50q - C(q)) = -\infty.$$

Plugging the critical points into π, we get

$$\pi(1) = 50 - 59 = -9 \qquad \text{and} \qquad \pi(9) = 450 - 374 > 0.$$

This is enough to show that profit is maximized at $q = 9$, that is, when Carson produces and sells 900 vacuum cleaners.

Answer: 9 hundred vacuum cleaners.

d. [2 points] Unsure how to solve the calculus problem in part **c.**, Carson just decides to produce and sell as many vacuum cleaners as he can. Unfortunately, a court order terminates Carson's business immediately after he had produced and sold 600 vacuum cleaners. At this point, had Carson's business *gained* or *lost* money? How much?

Give your answer by circling GAINED *or* LOST *and writing a* positive *number on the blank.*

Solution: Selling 600 vacuum cleaners at \$50 per unit nets Carson \$30,000. On the other hand, the cost of selling 600 vacuum cleaners is $C(6) = 251$ hundred dollars, or \$25,100. Thus Carson has gained $30000 - 25100 = 4900$ dollars after selling 600 vacuum cleaners.

Answer: Carson's business $\boxed{\text{GAINED}}$ LOST 49 hundred dollars.

8. [11 points] Consider the family of functions $f(x) = x^4 e^{-cx}$, where $c > 0$. Note that

$$f'(x) = x^3 e^{-cx}(4 - cx) \qquad \text{and} \qquad f''(x) = c^2 x^2 e^{-cx}\left(x - \tfrac{2}{c}\right)\left(x - \tfrac{6}{c}\right).$$

a. [2 points] List all the critical points of $f(x)$ and $f'(x)$, in terms of c. *No justification necessary.*

Answer: Critical points of $f(x)$: $\mathbf{0}, \frac{4}{c}$. Critical points of $f'(x)$: $\mathbf{0}, \frac{2}{c}, \frac{6}{c}$.

b. [4 points] Determine, in terms of the parameter c, the intervals of concavity of the function $f(x)$. Use calculus to find your answers, show all your work, and be sure to show enough evidence to justify your conclusions.

Solution: The function $f(x)$ will be concave up on intervals where $f''(x) > 0$, and concave down on intervals where $f''(x) < 0$. Noting that $f''(x) = 0$ at $x = 0, \frac{2}{c}, \frac{6}{c}$, we will make a sign chart for $f''(x)$ where we break up our number line at these three points and determine the sign of $f''(x)$ in each of the resulting intervals. Note that $c^2x^2e^{-cx} > 0$ for all $x \neq 0$, and that $(x - \frac{2}{c}) > 0$ when $x > \frac{2}{c}$, while $(x - \frac{6}{c}) > 0$ when $x > \frac{6}{c}$. This gives us the table of signs:

	$(-\infty, 0)$	$(0, \frac{2}{c})$	$(\frac{2}{c}, \frac{6}{c})$	$(\frac{6}{c}, \infty)$
$c^2x^2e^{-cx}$	+	+	+	+
$(x - \frac{2}{c})$	−	−	+	+
$(x - \frac{6}{c})$	−	−	−	+
$f''(x)$	+	+	−	+

Or, equivalently, the following sign chart:

$f''(x)$: $+ \cdot - \cdot - = +$ | 0 | $+ \cdot - \cdot - = +$ | $\frac{2}{c}$ | $+ \cdot + \cdot - = -$ | $\frac{6}{c}$ | $+ \cdot + \cdot + = +$

$\smile$ 0 $\smile$ $\frac{2}{c}$ $\frown$ $\frac{6}{c}$ $\smile$

Answer: Intervals on which $f(x)$ is concave up: $(-\infty, \frac{2}{c})$ **and** $(\frac{6}{c}, \infty)$

Answer: Intervals on which $f(x)$ is concave down: $(\frac{2}{c}, \frac{6}{c})$

c. [2 points] Circle the x-coordinates of all inflection points of $f(x)$ in terms of the parameter c that are listed below, or, if no inflection points of $f(x)$ are listed below, circle NONE.

0 e^{-c} e^{c} $\boxed{\frac{2}{c}}$ $\frac{4}{c}$ $\boxed{\frac{6}{c}}$ NONE

d. [3 points] For each $c > 0$, the function $y = f(x)$ has exactly one local extremum in $(0, \infty)$.

i. Find the unique value of c such that $f(x)$ has a local extreme value of $y = 1$ in the interval $(0, \infty)$. *Show your work.*

Solution: We are given that $f(x)$ has exactly one local extremum in $(0, \infty)$, so this extremum must occur at the only critical point $f(x)$ has in $(0, \infty)$, namely $x = \frac{4}{c}$. So we set $f\left(\frac{4}{c}\right) = 1$ and solve for c. We get

$$1 = f\left(\tfrac{4}{c}\right) = \left(\tfrac{4}{c}\right)^4 e^{-4}, \qquad \text{so} \quad e^4 = \left(\tfrac{4}{c}\right)^4, \qquad \text{which means} \quad e = \tfrac{4}{c}.$$

It follows that $c = \frac{4}{e}$. Since $f''\left(\frac{4}{c}\right) < 0$, this is a local MAX by the Second Derivative Test.

Answer: $c =$ $4/e$

ii. Is the local extremum of $f(x)$ in $(0, \infty)$ a max or a min? Circle your answer: $\boxed{\text{MAX}}$ MIN

(No justification is necessary.)

9. [10 points] A new internet meme has gone viral, and you are attempting to quantify its popularity. Let $V(m)$ model the total number of people who have *seen* the meme, and $L(m)$ the number of people who currently *like* the meme, m days after the meme first appeared. Assume the functions $V(m)$ and $L(m)$ have continuous derivatives, and that $V(m)$ is increasing. Also assume $L(0) = V(0) = 0$ and that everyone who likes the meme has seen it, but that people can change their minds over time about whether they like the meme.

a. [2 points] Circle the **one best** practical interpretation of the equation

$$\int_0^7 V'(m)\,dm \;=\; L(7).$$

$\boxed{i.}$ A week after the meme first appeared, everyone who has seen it likes it.

ii. The number of people who like the meme 7 days after it first appeared is the average rate of change of $V(m)$ over that week.

iii. The average number of new people per day who saw the meme over the first week since it appeared is equal to the number of people who liked it at the end of that week.

iv. The integral of the rate at which new people saw the meme over the first week since it appeared is equal to the net change in the number of new people who saw it that week.

Note: each of the parts **b.–d.** *below has at least one correct answer.*

b. [2 points] Circle **all equations** below that support the following statement: "Three weeks after the meme first appeared, about one new person is seeing it every second!"

(i) $V'(3) = 1$

$\boxed{\text{(ii)}}$ $V'(21) = 24 \cdot 60^2$

$\boxed{\text{(iii)}}$ $(V^{-1})'\big(V(21)\big) = \frac{1}{24\cdot 60^2}$

(iv) $(V^{-1})'(24 \cdot 60^2) = \frac{1}{V'(21)}$

c. [3 points] Circle **all** statements below that *must be true* if $\displaystyle\int_{20}^{27} L'(m)\,dm = -7$ million.

$\boxed{i.}$ The number of people who liked the meme 20 days after it first appeared was greater than the number of people who liked the meme 27 days after it first appeard.

$\boxed{ii.}$ At least one person who had previously liked the meme later changed their mind and decided they *didn't* like it.

$\boxed{iii.}$ There is some number m for which $L'(m) = -1{,}000{,}000$.

$\boxed{iv.}$ Twenty days after the meme first appeared, at least 7 million people had seen it.

d. [3 points] One noted expert in meme popularity argues that opinions change and memes fade over time, so she measures meme popularity by awarding one "popularity point" for *each day that one person likes a meme.* For example, one person liking a meme for three days would generate 3 popularity points, as would six people liking the meme for half a day each.

Circle **all** expressions below that give the total number of "popularity points" our meme would receive in the first three weeks after it first appeared, according to this expert's model.

(i) $L(21)$

(ii) $21 \cdot L(21)$

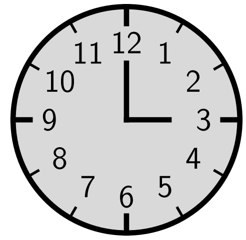
(iii) $\frac{L(21)-L(0)}{21-0}$

$\boxed{\text{(iv)}}$ $\int_0^{21} L(t)\,dt$

(v) $\int_0^{21} L'(t)\,dt$

(vi) $\frac{1}{21}\int_0^{21} L'(t)\,dt$

10. [8 points]



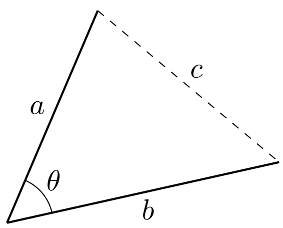
The hour and minute hands of a certain clock are 3 and 4 inches long, respectively. Suppose the hands move continuously in the clockwise direction, with the minute hand making one complete revolution every hour and the hour hand making one complete revolution every 12 hours.

a. [2 points] If θ is the radian measure of the (smaller) angle between the two hands, what is $\frac{d\theta}{dt}$ at 3pm, in radians per hour?

Solution: Since the minute hand moves at 2π radians per hour and the hour hand moves at $\frac{\pi}{6}$ radians per hour, at 3pm the angle between them is changing at a rate of

$$\frac{d\theta}{dt} = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6} \text{ radians per hour.}$$

Answer: $\frac{d\theta}{dt} =$ $-\frac{11\pi}{6}$



In part **b.**, you may want to use the *Law of Cosines*, which states that in a triangle where θ is the angle between two sides of length a and b, the length c of the third side satisfies the equation $c^2 = a^2 + b^2 - 2ab\cos\theta$. (See the picture to the left.)

b. [5 points] At what rate is the distance between the tips of the hour and minute hands on the clock changing at 3pm? *Show all of your work, and include units.*

Solution: Let z be the distance between the tips of the hour and minute hands, so

$$z^2 = 3^2 + 4^2 - (2)(3)(4)\cos\theta = 25 - 24\cos\theta.$$

Differentiating with respect to t, we get

$$2z\frac{dz}{dt} = 24\sin\theta\frac{d\theta}{dt}.$$

At 3pm we have $\theta = \frac{\pi}{2}$, so $\sin\theta = 1$, and $z^2 = 3^2 + 4^2$, so $z = 5$. Also we know $\frac{d\theta}{dt} = -\frac{11\pi}{6}$ from part **a**. Substituting these values gives us

$$2 \cdot 5 \cdot \frac{dz}{dt} = 24 \cdot 1 \cdot \frac{11\pi}{-6},$$

so

$$\frac{dz}{dt} = \frac{24 \cdot 11\pi}{-2 \cdot 5 \cdot 6} = -\frac{22\pi}{5}.$$

Thus the distance is *decreasing* at a rate of $\frac{22}{5\pi}$ inches per minute.

Answer: The distance is INCREASING $\boxed{\text{DECREASING}}$ at a rate of $\frac{22}{5\pi}$ **inches per minute** .

c. [1 point] Circle *all* times below at which the rate of change of the distance between the tips of the hour and minute hands is equal to zero, or else circle NONE OF THESE if there are none.

$\boxed{\text{12pm (noon)}}$ $\boxed{\text{6pm}}$ 9pm NONE OF THESE