Math 115 — Second Midterm — November 11, 2024

Write your 8-digit UMID number very clearly in the box to the right, and fill out the information on the lines below.

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ____________________

Instructor Name: ________________________________ Section #: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are <u>not</u> allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	10	
2	7	
3	8	
4	4	
5	6	

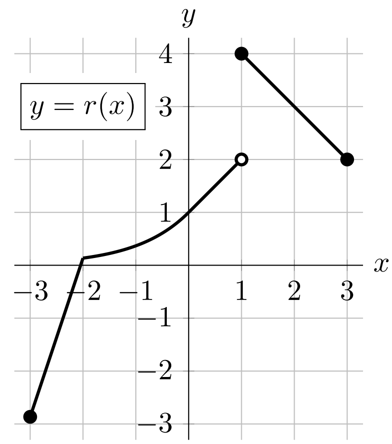
Problem	Points	Score
6	10	
7	13	
8	5	
9	7	
10	10	
Total	80	

1. [10 points]

The graph of the invertible function $r(x)$ defined on $[-3,3]$ is shown to the right. Note that:

- $r(x)$ is linear on $[0,1)$ and $[1,3]$;
- $r(x)$ is also linear, with slope 3, on $[-3,-2]$;
- $r(x) = e^x$ on $[-2,0]$; and

For parts **a.–d.**, find the **exact** values, or write DNE if the value does not exist. Your answers should not include the letter r but you do not need to simplify.



a. [2 points] Let $m(x) = \sqrt{2x} \cdot r(x)$. Find $m'(2)$.

Answer: $m'(2) =$ ____________________

b. [2 points] Let $q(x) = \dfrac{r(x)}{x}$. Find $q'(-1)$.

Answer: $q'(-1) =$ ____________________

c. [2 points] Let $w(x) = r\big(2e^x + \sin(3x)\big)$. Find $w'(0)$.

Answer: $w'(0) =$ ____________________

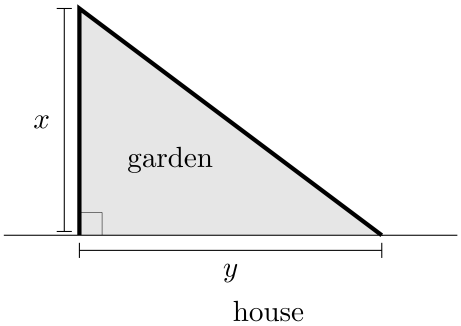
d. [2 points] Let $L(x) = r^{-1}(x)$. Find $L'(-2)$.

Answer: $L'(-2) =$ ____________________

e. [2 points] On which of the following intervals does $r(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers.

$[-3,-1]$ $\quad$ $[-2,0]$ $\quad$ $[-\frac{1}{2},\frac{1}{2}]$ $\quad$ $[0,1]$ $\quad$ $[0,2]$ $\quad$ NONE OF THESE

2. [7 points] Kieran is planning a small, fenced garden against his house. He wants the garden to be triangular as shown, but still needs to determine the lengths x and y, in meters. He has 10 meters of fencing to use. He only needs to fence two sides of the garden (the thick lines in the diagram), since his house will be the boundary on the third side.



a. [3 points] Find a formula for y in terms of x.

Answer: $y =$ ____________________

b. [2 points] Kieran plans to maximize the area of his garden. Find a formula for the function $G(x)$ giving the area of the garden in terms of x only. *Your formula should not include the letter y.*

Answer: $G(x) =$ ____________________

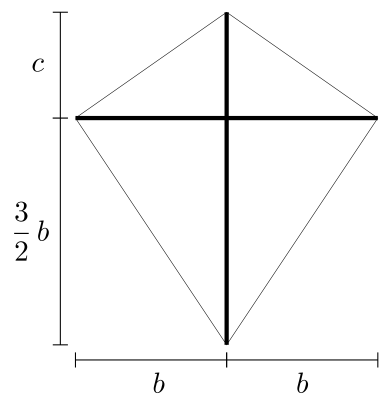
c. [2 points] What is the domain of $G(x)$ in the context of this problem?

Answer: ____________________

3. [8 points] Kamari is designing a kite. The kite's shape will be made of cloth, as shown below, and it will have a frame made of wooden rod in a cross shape, shown as the dark lines. The area A of the kite is given by the following formula (which you do not need to verify).

$$A = \frac{3}{2}b^2 + bc$$

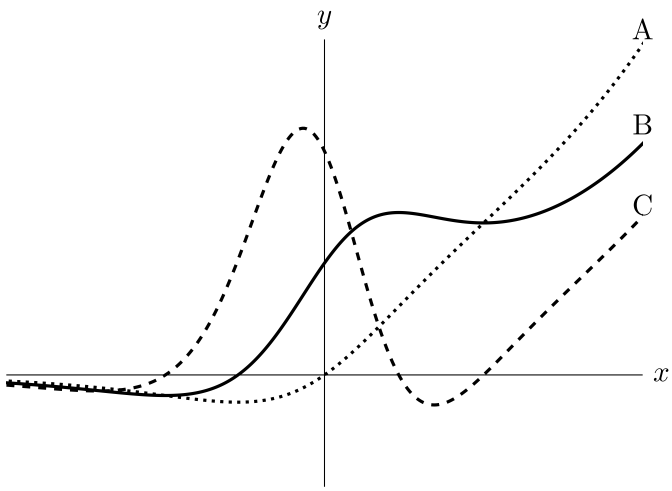
Kamari has 12 feet of wooden rod and wants to use all of it. What values of b and c will maximize the area of the kite? **Use calculus** to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the area of the kite.



Answer: The area of the kite is maximized when

$b =$ ____________________ $c =$ ____________________

4. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $f(x)$: __________

$f'(x)$: __________

$f''(x)$: __________

5. [6 points] The equation $\sin(x + y) = y^2$ defines y implicitly as a function of x.

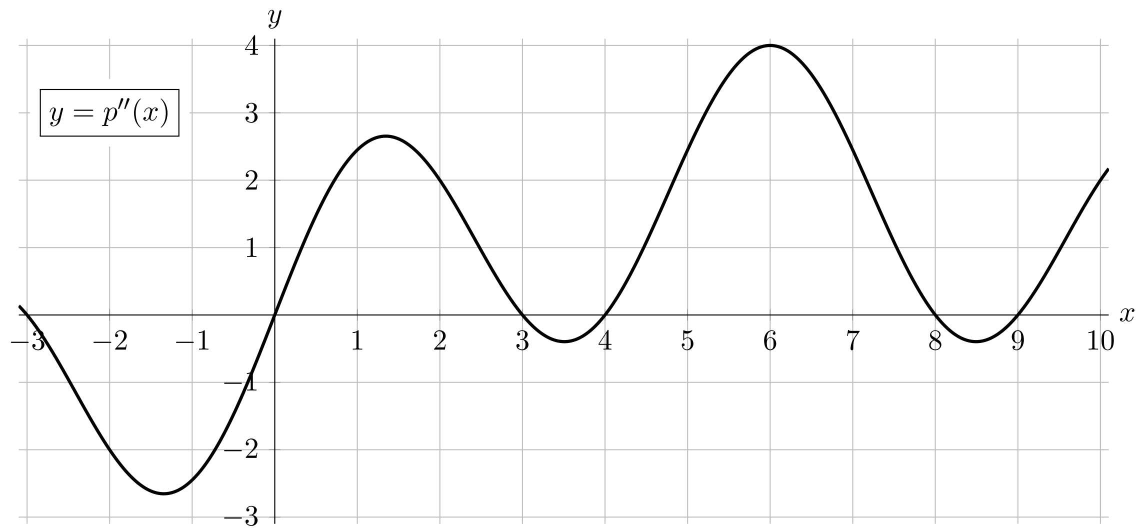
a. [4 points] Compute $\frac{dy}{dx}$. Show every step of your work.

Answer: $\frac{dy}{dx} =$ ____________________

b. [2 points] Find an equation of the line tangent to the curve defined by $\sin(x + y) = y^2$ at the point $(\pi, 0)$.

Answer: ____________________

6. [10 points] Suppose $p(x)$ is a function with continuous first and second derivatives that are defined for all real numbers. A portion of the graph of $p''(x)$, the **second derivative** of $p(x)$, is given below.



a. [2 points] Circle all intervals listed below on which $p(x)$ is concave up on the entire interval.

$(-2, 0)$ $(-1, 1)$ $(0, 2)$ $(3, 4)$ $(5, 7)$ NONE OF THESE

b. [2 points] Circle all intervals listed below on which $p'(x)$ is concave up on the entire interval.

$(-2, 0)$ $(-1, 1)$ $(0, 2)$ $(3, 4)$ $(5, 7)$ NONE OF THESE

c. [2 points] Circle all points listed below that are inflection points of $p(x)$.

$x = 0$ $x = 3$ $x = 3.5$ $x = 4$ $x = 6$ NONE OF THESE

d. [2 points] Circle all points below that are local minima of $p'(x)$.

$x = 0$ $x = 3$ $x = 3.5$ $x = 4$ $x = 6$ NONE OF THESE

e. [2 points] Assuming in this part that each of the x-values listed below is a critical point of $p(x)$, circle all points listed below that are local minima of $p(x)$.

$x = -2$ $x = 2$ $x = 3.5$ $x = 6$ $x = 8.5$ NONE OF THESE

7. [13 points] Let $g(x)$ be the piecewise function defined by

$$g(x) = \begin{cases} 1-(x+1)^{1/3} & x \le 0 \\ \dfrac{x^2-1.5x}{x-2} & x > 0 \text{ and } x \ne 2. \end{cases}$$

Note that $g(x)$ is defined and continuous everywhere except for $x = 2$, and that the derivative of $g(x)$ is given for *positive* inputs $x \ne 2$ by

$$g'(x) = \frac{x^2-4x+3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}, \qquad x > 0 \text{ and } x \ne 2.$$

a. [3 points] Find all critical points of the piecewise function $g(x)$.

Answer: $x =$ ______________________

b. [6 points] Find the x-coordinates of all local extrema of $g(x)$, and classify each as a local *maximum* or a local *minimum*. You must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

Answer: Local min(s) at $x =$ ______________________

Answer: Local max(es) at $x =$ ______________________

7. (continued) Recall from the last page that $g(x)$ is the piecewise function defined by

$$g(x) = \begin{cases} 1-(x+1)^{1/3} & x \leq 0 \\ \dfrac{x^2-1.5x}{x-2} & x > 0,\ x \neq 2. \end{cases}$$

Note that $g(x)$ is defined and continuous everywhere except for $x = 2$, and that the derivative of $g(x)$ is given for *positive* inputs $x \neq 2$ by

$$g'(x) = \frac{x^2-4x+3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}, \qquad x > 0 \text{ and } x \neq 2.$$

c. [4 points] Find the x-values where the global extrema of $g(x)$ occur on the interval $[-9, 1]$. Be sure to show your work and justify your answers.

Answer: The maximum occurs at $x =$ ____________________

Answer: The minimum occurs at $x =$ ____________________

8. [5 points] Suppose $h(t)$ is a continuous function with domain $[0, \infty)$ that also has the following properties:

- $\lim_{t\to\infty} h(t) = 10$.
- $h(t)$ has a local maximum at $t = 9$.
- $h(t)$ has exactly one critical point in the interval $(0, \infty)$.

Circle the numeral of each statement below that **must** be true.

i. $h'(1) > 0$.

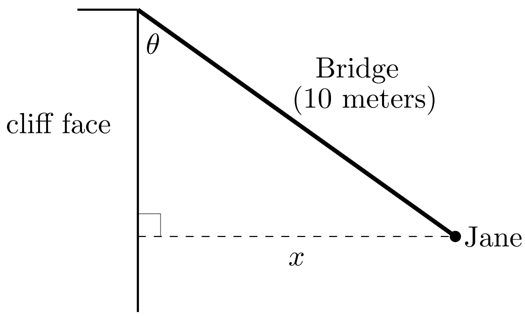
ii. $h'(9) = 0$.

iii. $h(t)$ has a global maximum at $t = 9$.

iv. $h(9) > 10$.

v. $h(t)$ has no global minimum on the interval $[0, \infty)$.

9. [7 points] Adventurer and treasure hunter "Michigan Jane" is running from bandits trying to take her loot! Just as Jane steps onto a bridge, the side she is standing on breaks, and the bridge swings toward the cliff face on the other side as shown in the figure below.



Note that the bridge is rigid and is 10 meters long. Assume that 2 seconds after the bridge snaps Jane is 8 meters from the cliff face (i.e. $x = 8$ in the figure above) and at that same moment the angle θ between the bridge and the cliff is decreasing at a rate of 2 radians per second.

Find the speed at which Jane is moving toward the cliff (i.e. her horizontal speed) 2 seconds after the bridge breaks. Show all of your work and give your answer in exact form, **with units**.

Answer: ______________________

10. [10 points] Tanya is riding her toboggan down the largest sledding hill in Ann Arbor! Assume that the height $H(x)$ of the hill above its base, in **hundreds** of feet, is given as a function of x, the **horizontal** distance from the top of the hill measured in **hundreds** of feet, where

$$H(x) = 1 - \frac{1}{(x-3)^2}, \quad 0 \le x \le 2.$$

a. [3 points] Write a formula for the linear approximation $L(x)$ of $H(x)$ near $x = 1$.

Answer: $L(x) =$ ____________________

b. [1 point] Use your formula from part **a.** to approximate the height of the hill at a **horizontal** distance of 90 feet from its top.

Answer: ____________________

c. [3 points] Assume that, as the toboggan goes down the hill, only its midpoint touches the hill. When the midpoint of the toboggan touches the hill at $x = 1$, the highest point of the toboggan is 76 feet above the base of the hill. Find the length of the toboggan in feet.

Answer: ____________________

d. [3 points] What is the x-value of the point on the hill where the angle between the toboggan and a horizontal line is 45 degrees? Note that $\tan(45^o) = 1$.

Answer: $x =$ ____________________