Math 115 — First Midterm — September 30, 2024

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 7 pages including this cover. There are 8 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	9	
3	9	
4	11	

Problem	Points	Score
5	6	
6	5	
7	5	
8	6	
Total	60	

1. [9 points] Ecologists are testing the water in a local wetland during three weeks of heavy rainfall in the late summer. They test for acidity (pH), temperature (T), and dissolved oxygen content (DO), making one measurement per week at the same time and location each week.

To the right is a table of their measurements in week w of the water's pH, temperature T in degrees Celsius, and DO in milligrams per liter. Unfortunately, their pH measurement in week 2 was faulty and had to be discarded. Use the values in the table to answer the questions below.

w	1	2	3
pH	9	?	7
T	25	25	23
DO	6.5	7.5	8.5

a. [1 point] Based on the given data, could T be a linear function of w, an exponential function of w, or neither? *Circle the one correct answer below.*

COULD BE LINEAR COULD BE EXPONENTIAL **[COULD NOT BE EITHER]**

b. [1 point] Based on the given data, could DO be a linear function of w, an exponential function of w, or neither? *Circle the one correct answer below.*

[COULD BE LINEAR] COULD BE EXPONENTIAL COULD NOT BE EITHER

c. [2 points] Assuming pH is a linear function of w, find a formula $L(w)$ for it.

Solution: To find the equation for the linear function L, we can use the point-slope form, i.e.

$$L(w) - 9 = m(w - 1),$$

where the slope $m = (7 - 9)/(3 - 1) = -1$. So, $L(w) = 9 - (w - 1) = 10 - w$.

Answer: $L(w) =$ $10 - w$

d. [3 points] Assuming pH is an exponential function of w, find a formula $E(w)$ for it.

Solution: Let us use the formula for the exponential function

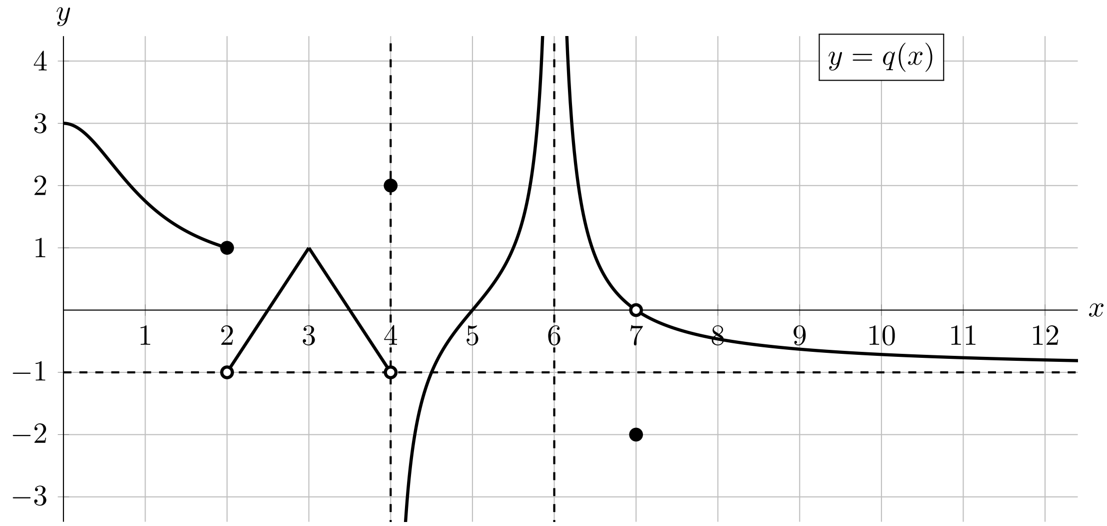
$$E(w) = E_0 a^w, \quad \text{where} \quad a = \left(\frac{7}{9}\right)^{\frac{1}{3-1}} = \frac{\sqrt{7}}{3} \quad \text{and} \quad E_0 = E(0) = \frac{E(1)}{a} = \frac{9}{\sqrt{7}/3} = \frac{27}{\sqrt{7}}.$$

Answer: $E(w) =$ $\frac{27}{\sqrt{7}}\left(\frac{\sqrt{7}}{3}\right)^w$

e. [2 points] Let $L(w)$ be the linear function from part **c.**, and $E(w)$ the exponential function from part **d.** above. **For each of the three pairs of values listed below, circle the value that is larger.**

$L(0)$ OR **[$E(0)$]** **[$L(2)$]** OR $E(2)$ $L(4)$ OR **[$E(4)$]**

2. [9 points] Below is a portion of the graph of the **even** function $q(x)$. Note that $q(x)$ is linear on the intervals $(2, 3)$ and $(3, 4)$, and has vertical asymptotes at $x = 4$ and $x = 6$ and a horizontal asymptote at $y = -1$.



a. [1 point] At which of the following values of x is the function $q(x)$ continuous? Circle all correct answers.

$x = 2$ $\boxed{x = 3}$ $\boxed{x = 5}$ $x = 6$ $x = 7$ NONE OF THESE

b. [6 points] Find the **exact** numerical value of each expression below, if possible. For any values that do not exist, including if they are limits that diverge to $\pm\infty$, write DNE. If there is not enough information to find a given value or determine whether it exists, write NEI. Remember that $q(x)$ is an **even** function, and $\pi \approx 3.14$.

i. $\lim_{x\to 7} q(x) =$ 0

ii. $\lim_{x\to 2} q(x) =$ **DNE**

iii. $\lim_{h\to 0} \frac{q(\pi + h) - q(\pi)}{h} =$ -2

iv. $\lim_{x\to -2^+} q(x) =$ 1

v. $\lim_{x\to 2^+} q(x^2 - 4) =$ 3

vi. $\lim_{x\to -\infty} \left(7q\left(\frac{x}{3}\right) + 1\right) =$ -6

c. [2 points] Given that $q'(5) = 1.5$, find an equation of the line tangent to the graph of $f(x) = q(x) - 4$ at the point $(5, -4)$.

Solution: The equation of the tangent line to the graph of $f(x)$ at the point (5,-4) has a form

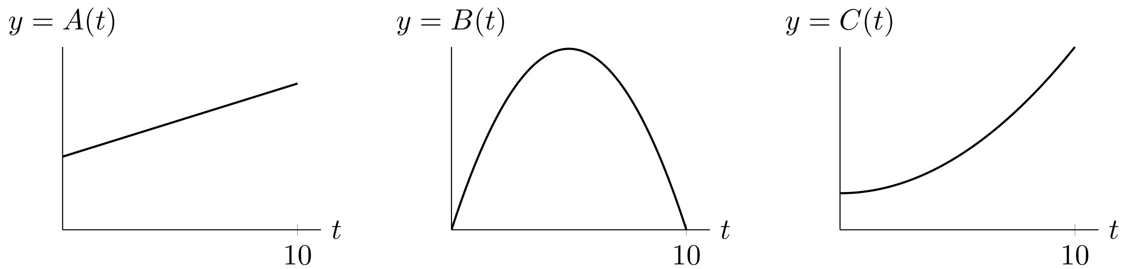
$$y - f(5) = f'(5)(x - 5).$$

Now we can plug in $f'(5) = q'(5) = 1.5$ and $f(5) = -4$.

Answer: $y =$ $-4 + 1.5(x - 5)$

3. [9 points] A few friends are biking on a long, straight trail.

a. [5 points] The functions $A(t)$, $B(t)$, and $C(t)$ give the distances of Ani, Bo, and Cait respectively from the start of the trail at time t minutes after 1:00pm for $0 \le t \le 10$. The vertical scales of the graphs are the same.



Answer the following. You may use A, B, and C to denote Ani, Bo, and Cait respectively.

i. Which friend has the greatest velocity at 1:01pm? **B**

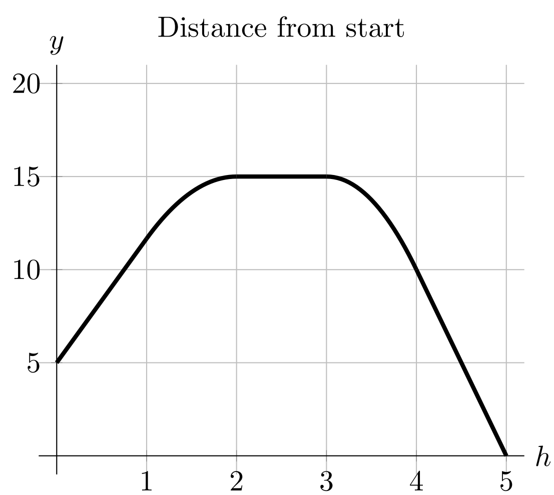
ii. Which friend has the greatest average velocity on $[0, 10]$? **C**

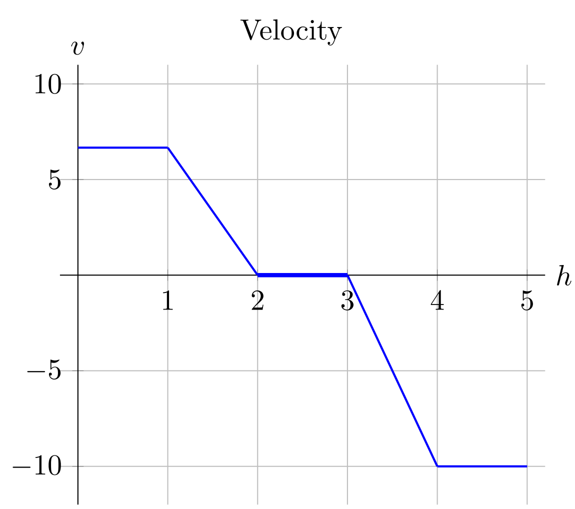
iii. Which friend is traveling at a constant speed? **A**

iv. At 1:10pm, which friend is furthest from where they were at 1pm? **C**

v. Which friend biked the greatest total distance on $[0, 10]$? **B**

b. [4 points] Another friend, Diego, sets off on a long ride. Diego's distance y from the start of the trail, in miles, h hours after 1:00pm is given below to the left. Note that the graph is linear on $[0, 1]$, $[2, 3]$, and $[4, 5]$. On the axes below to the right, carefully sketch a graph of Diego's velocity over the course of his ride. Be sure that your graph carefully indicates where his velocity is zero, positive, and negative, and where it is is increasing, decreasing, and constant.





4. [11 points] A city is replacing carbon-based energy production with solar power, saving large amounts of carbon that would otherwise have been burned in energy production. As a consultant for the city, you have created the following functions, which are both differentiable and invertible, to model these quantities.

- $E(t)$ is the solar capacity, in megawatts (MW), of the city t years after January 1, 2000.
- $C(s)$ is the amount of carbon, in tons, the city would save each day with a solar capacity of s MW.

a. [2 points] Using a complete sentence, give a practical interpretation of the equation

$$C^{-1}(15) = 3.$$

Solution: *In order to save 15 tons of carbon each day, the city needs a solar capacity of 3 MW.*

b. [3 points] Write a single equation involving E, C, and/or their inverses that represents the following statement.

At the start of 2008, the city saved twice as much carbon each day as it did at the start of 2001.

Answer: $C(E(8)) = 2C(E(1))$

c. [3 points] Write a single equation involving exactly one of the derivative functions E', C', $(E^{-1})'$ or $(C^{-1})'$ that represents the following statement.

If the city increased its solar capacity from 4.8 to 4.9 MW, it would save about 0.6 additional tons of carbon each day.

Answer: $C'(4.8) = 6$ **or** $C'(4.9) = 6$

d. [3 points] Using the fact that

$$(E^{-1})'(8) = \frac{2}{3},$$

answer the following question from the city, making sure you **include units** in your answer.

If we have a solar capacity of 8 MW at the start of 2025, approximately how much time do you expect it to take to increase our solar capacity an additional 0.5 MW?

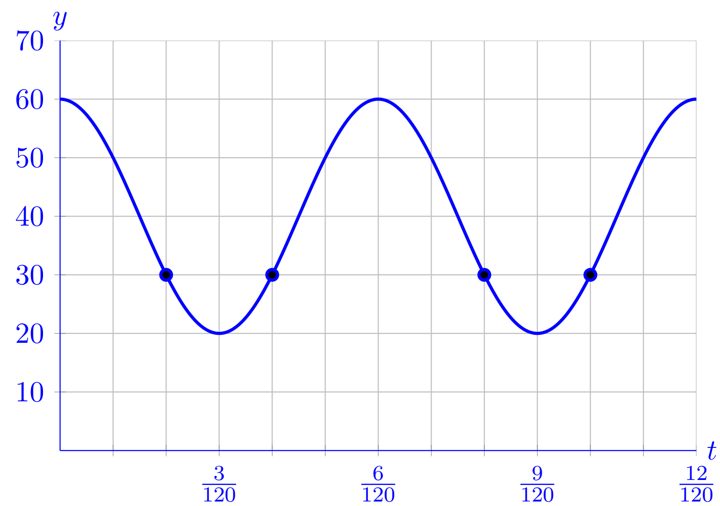
Answer: $\frac{1}{3}$ **of a year, or** 4 **months**

5. [6 points] The sound of a jet engine during aircraft take-off, measured at a distance of about 50 meters from the engine, is given by its sinusoidal sound pressure function $P(t)$, in Pascals (Pa), where t is the time in seconds after the plane enters the runway. The sound pressure reaches its maximum of 60 Pa at $t = 0$ seconds, and the smallest positive time where it reaches its minimum of 20 Pa is $t = 1/40$ seconds.

a. [3 points] Find a formula for $P(t)$.

Answer: $P(t) =$ $40 + 20\cos(40\pi t)$

b. [3 points] Given that the sound pressure function is 50 Pa at $t = 1/120$ seconds, find all t-values where $P(t)$ is equal to 30 Pa within the first 0.1 seconds.
(Note that $0.1 = \frac{2}{20} = \frac{4}{40}$*.)*



Answer: $t =$ $\frac{2}{120}, \frac{4}{120}, \frac{8}{120}, \frac{10}{120}$

6. [5 points] Let

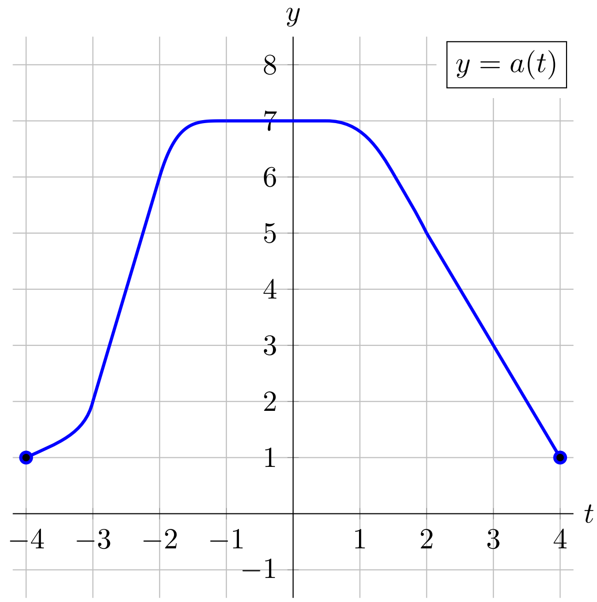
$$K(p) = 2^{\cos p} + \sqrt{(3p)}.$$

Use the limit definition of the derivative to write an explicit expression for $K'(5)$. *Your answer should not involve the letter K. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $K'(5) =$ $\lim_{h\to 0} \frac{2^{\cos(5+h)} + \sqrt{3(5+h)} - \left(2^{\cos 5} + \sqrt{15}\right)}{h}$

7. [5 points] Suppose $a(t)$ is the altitude in hundreds of meters above sea level of a certain hot air balloon t hours after 12pm noon on a sunny day. Carefully draw a plausible graph of $a(t)$ on the given axes, assuming the following are true:

- the balloon lifts off the ground at 8am from a point that is 100 meters above sea level, and stays in the air until it lands at the same location at 4pm;
- the rate at which the balloon's altitude is changing is constant between 9am and 10am, and again between 2pm and 4pm;
- at 9:30am, the balloon is **ascending** twice as fast as it is **descending** at 3pm;
- the balloon spends at least one full hour at its maximum altitude of 700 meters.



8. [6 points] Let $g(x)$ be the piecewise function defined by

$$g(x) = \begin{cases} \dfrac{-4(x+1)}{(x^2-1)(x+4)} & x < 0 \\ e^{A(x-1)} + \dfrac{B(x+1)^2(x-2)}{2(x-3)(x-2)^2} & x \geq 0 \end{cases}$$

where A and B are nonzero constants.

a. [3 points] List the x-coordinates of all **vertical asymptotes** of $g(x)$.

Answer: $x =$ $-4, 2, 3$

b. [3 points] Find values of the constants A and B such that $g(x)$ is continuous at $x = 0$ and $g(x)$ has a **horizontal asymptote** at $y = -3$.

Solution: We can get the horizontal asymptote at $y = -3$ if $A \leq 0$ and then

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{Bx^3}{2x^3} = B/2 = -3,$$

So $B = -6$. To make g continuous at 0, we set up the equation

$$\frac{-4(0+1)}{(0^2-1)(0+4)} = e^{A(0-1)} + \frac{-6(0+1)^2(0-2)}{2(0-3)(0-2)^2}, \quad \text{or} \quad 1 = e^{-A} - \frac{1}{2}.$$

Now solving for A by taking logarithms, we get

$$A = \ln(2/3).$$

Answer: $A =$ $\ln(2/3)$ and $B =$ -6