Math 115 — Second Midterm — November 11, 2024

EXAM SOLUTIONS

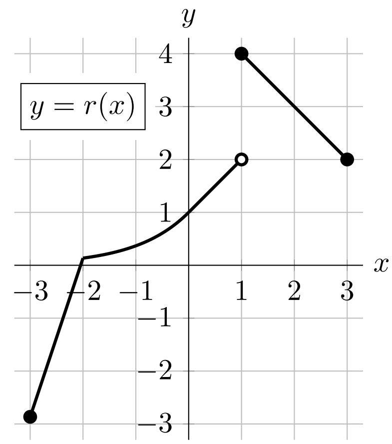
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 10 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	7	
3	8	
4	4	
5	6	

Problem	Points	Score
6	10	
7	13	
8	5	
9	7	
10	10	
Total	80	

1. [10 points]

The graph of the invertible function $r(x)$ defined on $[-3,3]$ is shown to the right. Note that:



- $r(x)$ is linear on $[0,1)$ and $[1,3]$;
- $r(x)$ is also linear, with slope 3, on $[-3,-2]$;
- $r(x) = e^x$ on $[-2,0]$; and

For parts **a.–d.**, find the **exact** values, or write DNE if the value does not exist. Your answers should not include the letter r but you do not need to simplify.

a. [2 points] Let $m(x) = \sqrt{2x} \cdot r(x)$. Find $m'(2)$.

Solution:

$$m'(2) = \frac{1}{2}(2\cdot 2)^{-1/2}\cdot 2\cdot r(2) + \sqrt{2\cdot 2}\cdot r'(2) = \frac{r(2)}{2} + 2r'(2) = \frac{3}{2} + 2(-1) = -\frac{1}{2}.$$

Answer: $m'(2) =$ $-1/2$

b. [2 points] Let $q(x) = \dfrac{r(x)}{x}$. Find $q'(-1)$.

Solution:

$$q'(-1) = \frac{r'(-1)\cdot(-1) - r(-1)\cdot 1}{(-1)^2} = \frac{-e^{-1} - e^{-1}}{1} = -2e^{-1}.$$

Answer: $q'(-1) =$ $-2/e$

c. [2 points] Let $w(x) = r\big(2e^x + \sin(3x)\big)$. Find $w'(0)$.

Solution: $w'(x) = r'(2e^x + \sin 3x)\cdot(2e^x + 3\cos 3x)$, so

$$w'(0) = r'(2)\cdot(2+3) = (-1)\cdot 5 = -5.$$

Answer: $w'(0) =$ -5

d. [2 points] Let $L(x) = r^{-1}(x)$. Find $L'(-2)$.

Solution: Since $r(x)$ is linear on $[-3,-2]$ with slope 3, and $-3 \le r^{-1}(-2) \le -2$, we have

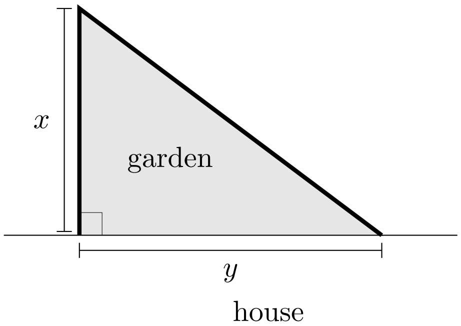
$$(r^{-1})'(-2) = \frac{1}{r'\big(r^{-1}(-2)\big)} = \frac{1}{3}.$$

Answer: $L'(-2) =$ $1/3$

e. [2 points] On which of the following intervals does $r(x)$ satisfy the hypotheses of the Mean Value Theorem? Circle all correct answers.

$[-3,-1]$ $\quad$ **[$[-2,0]$]** $\quad$ **[$[-\frac{1}{2},\frac{1}{2}]$]** $\quad$ $[0,1]$ $\quad$ $[0,2]$ $\quad$ NONE OF THESE

2. [7 points] Kieran is planning a small, fenced garden against his house. He wants the garden to be triangular as shown, but still needs to determine the lengths x and y, in meters. He has 10 meters of fencing to use. He only needs to fence two sides of the garden (the thick lines in the diagram), since his house will be the boundary on the third side.



a. [3 points] Find a formula for y in terms of x.

Solution: By the Pythagorean Theorem, the hypotenuse of the triangle has length $\sqrt{x^2+y^2}$, so the total length of fencing is

$$x + \sqrt{x^2+y^2} \;=\; 10.$$

Now we solve for y, keeping in mind that y should be nonnegative:

$$\begin{aligned} \sqrt{x^2+y^2} &= 10 - x \\ x^2+y^2 &= (10-x)^2 \\ y^2 &= (10-x)^2 - x^2 \\ y &= \sqrt{(10-x)^2 - x^2} \;=\; \sqrt{100-20x}. \end{aligned}$$

Answer: $y =$ $\sqrt{(10-x)^2 - x^2}$ **or** $\sqrt{100-20x}$

b. [2 points] Kieran plans to maximize the area of his garden. Find a formula for the function $G(x)$ giving the area of the garden in terms of x only. *Your formula should not include the letter y.*

Solution: The area of the garden is

$$G(x) \;=\; \frac{1}{2}xy \;=\; \frac{1}{2}x\sqrt{100-20x}.$$

Answer: $G(x) =$ $\frac{1}{2}x\sqrt{100-20x}$

c. [2 points] What is the domain of $G(x)$ in the context of this problem?

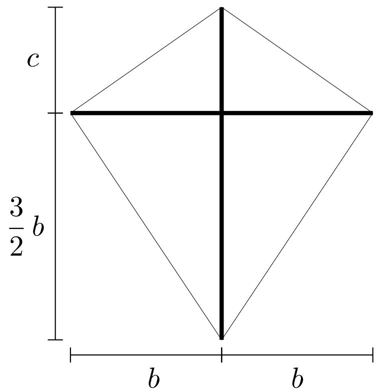
Solution: In order for the garden to have the triangular shape given in the figure, both x and y must be positive. Since the hypotenuse of a right triangle must be longer than either of the legs, we must have $x < 5$. Therefore, since any $0 < x < 5$ corresponds to a right triangle as shown in the figure, the domain of $G(x)$ is $(0, 5)$. (We will not be picky about endpoints, and also accept $[0, 5]$.)

Answer: $(0, 5)$ **or** $[0, 5]$

3. [8 points] Kamari is designing a kite. The kite's shape will be made of cloth, as shown below, and it will have a frame made of wooden rod in a cross shape, shown as the dark lines. The area A of the kite is given by the following formula (which you do not need to verify).

$$A = \frac{3}{2}b^2 + bc$$

Kamari has 12 feet of wooden rod and wants to use all of it. What values of b and c will maximize the area of the kite? **Use calculus** to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the area of the kite.



Solution: The total length of wooden rod used in constructing the kite is

$$b + b + \frac{3}{2}b + c \;=\; \frac{7}{2}b + c.$$

Therefore, we want to maximize the kite's area,

$$A \;=\; \frac{3}{2}b^2 + bc, \qquad (*)$$

subject to the constraint $\frac{7}{2}b + c = 12$. Using the constraint equation to solve for c in terms of b, we get $c = 12 - \frac{7}{2}b$. Substituting this for c in $(*)$ above, we get

$$A \;=\; \frac{3}{2}b^2 + b\left(12 - \frac{7}{2}b\right) \;=\; 12b - 2b^2.$$

In the context of the problem, b must be positive and c nonnegative, so $12 - \frac{7}{2}b = c \geq 0$, which means $0 < b \leq \frac{24}{7}$. So we want to maximize $12b - b^2$ on the domain $0 < b \leq \frac{24}{7}$.
The function $A = 12b - 2b^2$ is a parabola with negative leading coefficient, so its maximum will occur at its vertex provided its vertex is within the given domain. Using calculus, we find the derivative $\frac{dA}{db}$, set $\frac{dA}{db}$ equal to zero, and solve for b. This gives us:

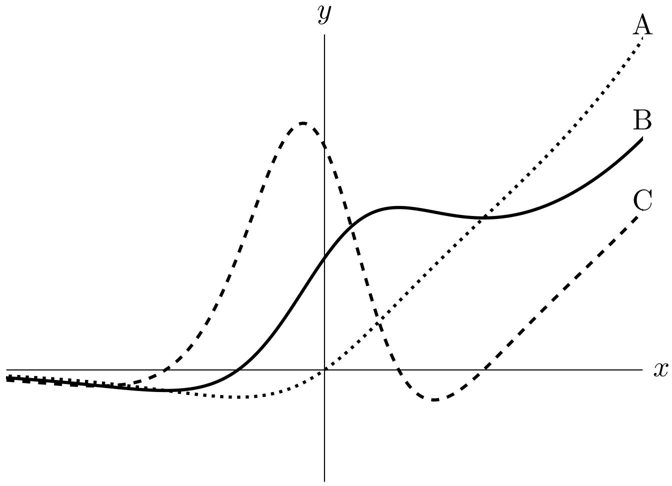
$$\frac{dA}{db} \;=\; 12 - 4b \;=\; 0 \qquad \text{when } b = 3.$$

Since $0 < 3 < \frac{24}{7}$, the vertex at $b = 3$ does occur within the domain, so it is the global maximum of A on the interval $0 < b \leq \frac{24}{7}$. Finally, by the constraint equation, when $b = 3$ we have $c = 12 - \frac{7}{2}(3) = \frac{3}{2}$.

Answer: The area of the kite is maximized when

$b =$ 3 $\qquad\qquad c =$ 3/2

4. [4 points] Shown below are portions of the graphs of the functions $y = f(x)$, $y = f'(x)$, and $y = f''(x)$. Determine which graph is which, and then, on the answer lines below, indicate after each function the letter A, B, or C that corresponds to its graph. No work or justification is needed.



Answer: $f(x)$: A

$f'(x)$: B

$f''(x)$: C

5. [6 points] The equation $\sin(x+y) = y^2$ defines y implicitly as a function of x.

a. [4 points] Compute $\dfrac{dy}{dx}$. Show every step of your work.

Solution: Implicitly differentiating the equation $\sin(x+y) = y^2$ with respect to x, we get

$$\begin{aligned}\cos(x+y)\left(1+\frac{dy}{dx}\right) &= 2y\frac{dy}{dx}\\ \cos(x+y)+\cos(x+y)\frac{dy}{dx} &= 2y\frac{dy}{dx}\\ \cos(x+y) &= 2y\frac{dy}{dx} - \cos(x+y)\frac{dy}{dx}\\ \frac{\cos(x+y)}{2y-\cos(x+y)} &= \frac{dy}{dx}.\end{aligned}$$

Answer: $\dfrac{dy}{dx} = \dfrac{\cos(x+y)}{2y-\cos(x+y)}$

b. [2 points] Find an equation of the line tangent to the curve defined by $\sin(x+y) = y^2$ at the point $(\pi, 0)$.

Solution: From the answer to part **a.**, we see the the derivative $\frac{dy}{dx}$ at the point $(x,y) = (\pi, 0)$ is

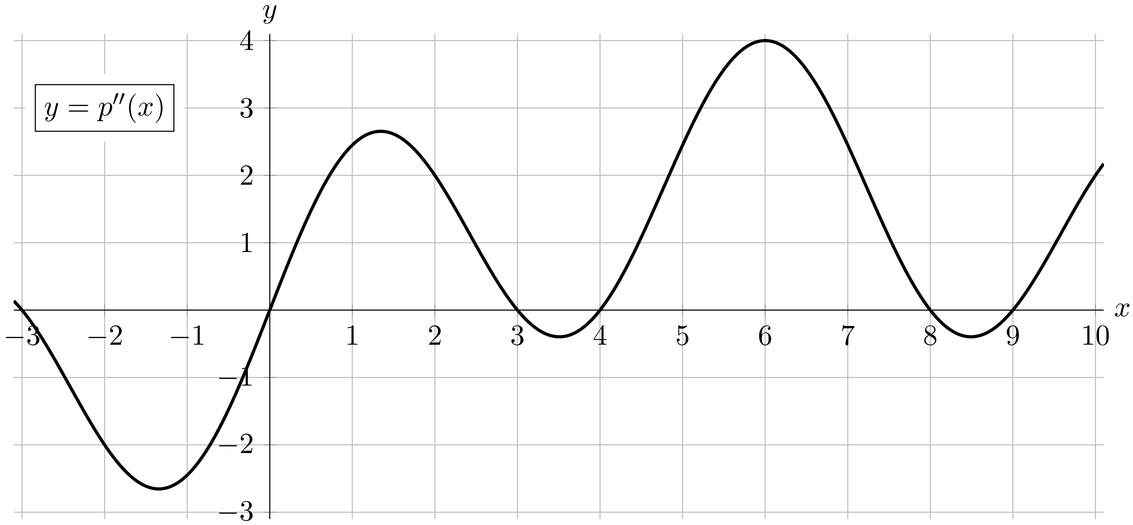
$$\left.\frac{dy}{dx}\right|_{(x,y)=(\pi,0)} = \frac{\cos(\pi+0)}{2(0)-\cos(\pi+0)} = \frac{\cos(\pi)}{-\cos(\pi)} = -1.$$

So we want an equation for the line that passes through the point $(\pi, 0)$ with a slope of -1. Using point-slope form, we get

$$y - 0 = -1(x-\pi), \qquad \text{or} \qquad y = \pi - x.$$

Answer: $y = \pi - x$

6. [10 points] Suppose $p(x)$ is a function with continuous first and second derivatives that are defined for all real numbers. A portion of the graph of $p''(x)$, the **second derivative** of $p(x)$, is given below.



a. [2 points] Circle all intervals listed below on which $p(x)$ is concave up on the entire interval.

$(-2, 0)$ $(-1, 1)$ $\boxed{(0, 2)}$ $(3, 4)$ $\boxed{(5, 7)}$ NONE OF THESE

b. [2 points] Circle all intervals listed below on which $p'(x)$ is concave up on the entire interval.

$(-2, 0)$ $\boxed{(-1, 1)}$ $(0, 2)$ $(3, 4)$ $(5, 7)$ NONE OF THESE

c. [2 points] Circle all points listed below that are inflection points of $p(x)$.

$\boxed{x = 0}$ $\boxed{x = 3}$ $x = 3.5$ $\boxed{x = 4}$ $x = 6$ NONE OF THESE

d. [2 points] Circle all points below that are local minima of $p'(x)$.

$\boxed{x = 0}$ $x = 3$ $x = 3.5$ $\boxed{x = 4}$ $x = 6$ NONE OF THESE

e. [2 points] Assuming in this part that each of the x-values listed below is a critical point of $p(x)$, circle all points listed below that are local minima of $p(x)$.

$x = -2$ $\boxed{x = 2}$ $x = 3.5$ $\boxed{x = 6}$ $x = 8.5$ NONE OF THESE

7. [13 points] Let $g(x)$ be the piecewise function defined by

$$g(x) = \begin{cases} 1-(x+1)^{1/3} & x \le 0 \\ \dfrac{x^2-1.5x}{x-2} & x > 0 \text{ and } x \neq 2. \end{cases}$$

Note that $g(x)$ is defined and continuous everywhere except for $x = 2$, and that the derivative of $g(x)$ is given for *positive* inputs $x \neq 2$ by

$$g'(x) = \frac{x^2-4x+3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}, \qquad x > 0 \text{ and } x \neq 2.$$

a. [3 points] Find all critical points of the piecewise function $g(x)$.

Solution: The critical points of $g(x)$ are points c in the domain of $g(x)$ such that $g'(c)$ is zero or does not exist.

For $x > 0$, we are given $g'(x) = \frac{(x-1)(x-3)}{(x-2)^2}$, so $g'(x) = 0$ when $x = 1$ or $x = 3$, and $g'(x)$ DNE when $x = 2$. So $x = 1$ and $x = 3$ are critical points of $g(x)$, but since $x = 2$ is not in the domain of $g(x)$, we do not include $x = 2$ as a critical point of $g(x)$.

For $x < 0$, we have $g'(x) = -\frac{1}{3}(x+1)^{-2/3}$, which is never zero but fails to exist when $x = -1$, so $x = -1$ is a critical point of $g(x)$.

Finally, we must check the point $x = 0$. Since

$$\lim_{x\to 0^+} g'(x) = \frac{3}{4} \qquad \text{and} \qquad \lim_{x\to 0^-} g(x) = -\frac{1}{3},$$

we see that $g'(0)$ does not exist, so $x = 0$ is a critical point of $g(x)$.

Answer: $x =$ **−1, 0, 1, 3**

b. [6 points] Find the x-coordinates of all local extrema of $g(x)$, and classify each as a local *maximum* or a local *minimum*. You must use calculus to find and justify your answers. Be sure your conclusions are clearly stated and that you show enough evidence to support them.

Solution: We apply the first derivative test to the critical points from part **a**. Since $g(x)$ is not defined at $x = 2$, $g'(x)$ could change sign at $x = 2$, so we include $x = 2$ in our number line as well. Note that $(x+1)^{-2/3}$ is positive for all $x < 0$, so $g'(x) = -\frac{1}{3}(x+1)^{-2/3}$ is negative for all $x < 0$. Using $g'(x) = \frac{(x-1)(x-3)}{(x-2)^2}$ for $x > 0$, this gives us the following sign chart for $g'(x)$:

	$x<-1$	$-1<x<0$	$0<x<1$	$1<x<2$	$2<x<3$	$x>3$
$g'(x)$:	$-$	$-$	$\frac{-\cdot -}{+} = +$	$\frac{+\cdot -}{+} = -$	$\frac{+\cdot -}{+} = -$	$\frac{+\cdot +}{+} = +$

By the First Derivative Test, it follows that $g(x)$ has local minima at $x = 0$ and $x = 3$, where $g'(x)$ changes from negative to positive, and a local maximum at $x = 1$, where $g'(x)$ changes from positive to negative.

Answer: Local min(s) at $x =$ **0, 3**

Answer: Local max(es) at $x =$ **1**

7. (continued) Recall from the last page that $g(x)$ is the piecewise function defined by

$$g(x) = \begin{cases} 1-(x+1)^{1/3} & x \le 0 \\ \dfrac{x^2-1.5x}{x-2} & x > 0,\ x \neq 2. \end{cases}$$

Note that $g(x)$ is defined and continuous everywhere except for $x = 2$, and that the derivative of $g(x)$ is given for *positive* inputs $x \neq 2$ by

$$g'(x) = \frac{x^2-4x+3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}, \qquad x > 0 \text{ and } x \neq 2.$$

c. [4 points] Find the x-values where the global extrema of $g(x)$ occur on the interval $[-9, 1]$. Be sure to show your work and justify your answers.

Solution: Since $g(x)$ is continuous on $[-9,1]$, we can just calculate the values of $g(x)$ at each critical point of $g(x)$ in $(-9,1)$, along with the values $g(-9)$ and $g(1)$, and identify the greatest and least of these values as the global max and min, respectively, of $g(x)$ on $[-9,1]$. By part **a.**, the critical points of $g(x)$ in $(-9,1)$ are $x=-1$ and $x=0$. Then we compute:

$$g(-9) = 1-(-8)^{1/3} = 1-(-2) = 3, \qquad g(-1) = 1, \qquad g(0) = 0, \qquad g(1) = \frac{1-1.5}{1-2} = \frac{1}{2}.$$

It follows that on $[-9,1]$, $g(x)$ has a global max of 3 occurring at $x=-9$, and a global min of 0 occurring at $x = 0$.

Answer: The maximum occurs at $x =$ -9

Answer: The minimum occurs at $x =$ 0

8. [5 points] Suppose $h(t)$ is a continuous function with domain $[0,\infty)$ that also has the following properties:

- $\lim_{t\to\infty} h(t) = 10$.
- $h(t)$ has a local maximum at $t = 9$.
- $h(t)$ has exactly one critical point in the interval $(0,\infty)$.

Circle the numeral of each statement below that **must** be true.

i. $h'(1) > 0$.

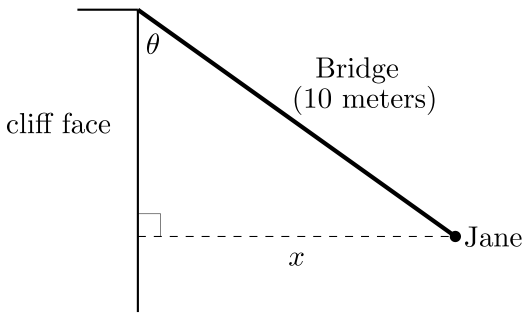
ii. $h'(9) = 0$.

iii. $h(t)$ has a global maximum at $t = 9$.

iv. $h(9) > 10$.

v. $h(t)$ has no global minimum on the interval $[0,\infty)$.

9. [7 points] Adventurer and treasure hunter "Michigan Jane" is running from bandits trying to take her loot! Just as Jane steps onto a bridge, the side she is standing on breaks, and the bridge swings toward the cliff face on the other side as shown in the figure below.



Note that the bridge is rigid and is 10 meters long. Assume that 2 seconds after the bridge snaps Jane is 8 meters from the cliff face (i.e. $x = 8$ in the figure above) and at that same moment the angle θ between the bridge and the cliff is decreasing at a rate of 2 radians per second.

Find the speed at which Jane is moving toward the cliff (i.e. her horizontal speed) 2 seconds after the bridge breaks. Show all of your work and give your answer in exact form, **with units**.

Solution: The variables θ and x are functions of time, t, and we are given that

$$x = 8 \text{ meters} \qquad \text{and} \qquad \frac{d\theta}{dt} = -2 \text{ radians per second}$$

when $t = 2$. We want to find $\frac{dx}{dt}$ when $t = 2$. So we will need an equation relating θ and x. Using trigonometry, we get the equation

$$\sin\theta = \frac{x}{10}.$$

Differentiating this equation with respect to t gives us

$$(\cos\theta)\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}. \qquad (*)$$

In order to find $\frac{dx}{dt}$ from $(*)$ when $t = 2$, we just need to know $\cos(\theta)$ when $t = 2$. For this we again use trigonometry to get

$$\cos(\theta)\Big|_{t=2} = \frac{y}{10},$$

where y is the length of the side of the triangle in the figure that is parallel to the cliff face, so $y = \sqrt{10^2 - 8^2} = 6$ when $t = 2$ by the Pythagorean Theorem. Now we can plug $\cos\theta = \frac{6}{10}$ and $\frac{d\theta}{dt} = -2$ into $(*)$ to get

$$\frac{dx}{dt}\Big|_{t=2} = 10 \cdot \frac{6}{10} \cdot (-2) = -12 \text{ meters per second.}$$

Answer: 12 **m/s**

10. [10 points] Tanya is riding her toboggan down the largest sledding hill in Ann Arbor! Assume that the height $H(x)$ of the hill above its base, in **hundreds** of feet, is given as a function of x, the **horizontal** distance from the top of the hill measured in **hundreds** of feet, where

$$H(x) = 1 - \frac{1}{(x-3)^2}, \quad 0 \le x \le 2.$$

a. [3 points] Write a formula for the linear approximation $L(x)$ of $H(x)$ near $x = 1$.

Solution: We have $H(1) = 1 - \frac{1}{(1-3)^2} = \frac{3}{4}$ and

$$H'(1) = 2(1-3)^{-3} = 2\left(-\frac{1}{8}\right) = -\frac{1}{4}.$$

Therefore

$$L(x) = H(1) + H'(1)(x-1) = \frac{3}{4} - \frac{1}{4}(x-1).$$

Answer: $L(x) =$ $\frac{3}{4} - \frac{1}{4}(x-1)$

b. [1 point] Use your formula from part **a.** to approximate the height of the hill at a **horizontal** distance of 90 feet from its top.

Solution: Since 90 feet is 0.9 hundred feet, we get the approximation

$$L(0.9) = \frac{3}{4} - \frac{1}{4}(0.9-1) = \frac{3}{4} - \frac{1}{4}(-0.1) = \frac{3}{4} + \frac{1}{40} = \frac{31}{40}.$$

Answer: $\frac{31}{40}$ **hundred feet**

c. [3 points] Assume that, as the toboggan goes down the hill, only its midpoint touches the hill. When the midpoint of the toboggan touches the hill at $x = 1$, the highest point of the toboggan is 76 feet above the base of the hill. Find the length of the toboggan in feet.

Solution: When the midpoint of the toboggan touches the hill at $x = 1$, the slope of the line parallel to the toboggan is given by $H'(1) = -\frac{1}{4}$, and the height above the base of the hill of the midpoint of the toboggan is $H(1) = \frac{3}{4}$ hundred feet, or 75 feet. This means the highest point of the toboggan, which will be its back tip, is 1 foot higher than its midpoint, and therefore 2 feet higher than its front tip. So the toboggan itself forms the hypotenuse of a right triangle of height 2 feet and slope $-\frac{1}{4}$. This triangle has width 8 feet, which makes the hypotenuse

$$\sqrt{2^2 + 8^2} = \sqrt{68} = 2\sqrt{17} \quad \text{feet long.}$$

Answer: $2\sqrt{17}$

d. [3 points] What is the x-value of the point on the hill where the angle between the toboggan and a horizontal line is 45 degrees? Note that $\tan(45^o) = 1$.

Solution: The angle between the toboggan and a horizontal line will be 45 degrees wherever the slope of the hill is ± 1. Since $H(x)$ is decreasing on $0 \le x \le 2$, this means we are looking for a point where $H'(x) = -1$. Solving $H'(x) = \frac{2}{(x-3)^3} = -1$, we get $x = 3 - 2^{1/3}$.

Answer: $x =$ $3 - 2^{1/3}$