Math 115 — Final Exam — December 12, 2024

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 11 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
   No other scratch paper is allowed, and any other scratch work submitted will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are not allowed to use a calculator of any kind on this exam.
   You are allowed notes written on two sides of a $3'' \times 5''$ note card.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	13	
2	15	
3	10	
4	4	
5	6	
6	10	

Problem	Points	Score
7	8	
8	9	
9	10	
10	8	
11	7	
Total	100	

1. [13 points] Given below is a table of values for the **decreasing** function $H(w)$ and its derivative, $H'(w)$. Suppose the functions $H(w)$, $H'(w)$, and $H''(w)$ are all defined and continuous on $(-\infty, \infty)$.

w	0	1	2	3	4	5	6	7	8	9	10
$H(w)$	15	13	12	10	9	7	6	5	3	1	0
$H'(w)$	0	-1	-5	-3	-3	-2	-1	0	-1	0	-2

a. [3 points] Use a right–hand Riemann sum with **five** equal subdivisions to estimate $\int_0^{10} H(w)\,dw$. Write out all the terms in your sum. You do not need to simplify, but your answer should not include the letter H.

Solution:

$$\begin{aligned} 12 \times 2 + 9 \times 2 + 6 \times 2 + 3 \times 2 + 0 \times 2 &= \\ 2\Big(12 + 9 + 6 + 3 + 0\Big) &= 60 \end{aligned}$$

b. [1 point] Does the answer to part **a.** overestimate, underestimate, or equal the value of $\int_0^{10} H(w)\,dw$? Circle your answer. If there is not enough information, circle NEI. You do not need to show any work for this part of the problem.

OVERESTIMATE $\qquad$ $\boxed{\text{UNDERESTIMATE}}$ $\qquad$ EQUAL $\qquad$ NEI

c. [2 points] How many equal subdivisions of $[0, 10]$ are needed so that the difference between the left and right Riemann sum approximations of $\int_0^{10} H(w)\,dw$ is exactly 1.5?

Solution:

$$\begin{aligned} \Big(H(10) - H(0)\Big)\frac{(b-a)}{n} &= R - L \\ \frac{(15-0)(10-0)}{n} &= 1.5 \\ n &= 100. \end{aligned}$$

Answer: 100

1. (continued) *The information from the problem is repeated for convenience.*

Given below is a table of values for the **decreasing** function $H(w)$ and its derivative, $H'(w)$. Suppose the functions $H(w)$, $H'(w)$, and $H''(w)$ are all defined and continuous on $(-\infty, \infty)$.

w	0	1	2	3	4	5	6	7	8	9	10
$H(w)$	15	13	12	10	9	7	6	5	3	1	0
$H'(w)$	0	-1	-5	-3	-3	-2	-1	0	-1	0	-2

In **d.–f.**, give numerical answers.

d. [2 points] Find the average rate of change of $H'(w)$ on the interval $[3, 7]$.

Solution:

$$\frac{H'(7) - H'(3)}{7 - 3} = \frac{0 - (-3)}{7 - 3} = \frac{3}{4}$$

Answer: 3/4

e. [2 points] Use the table to estimate $H''(1.5)$.

Solution:

$$\frac{H'(2) - H'(1)}{2 - 1} = \frac{-5 - (-1)}{2 - 1} = -4$$

Answer: -4

f. [3 points] Find $\displaystyle\int_2^5 \big(3H'(w) - 2w\big)\, dw$.

Solution:

$$\begin{aligned}
\int_2^5 \big(3H'(w) - 2w\big)\, dw &= 3\int_2^5 H'(w)\, dw - 2\int_2^5 w\, dw \\
&= 3\big(H(5) - H(2)\big) - w^2\Big|_2^5 \\
&= 3(7 - 12) - (5^2 - 2^2) = -36
\end{aligned}$$

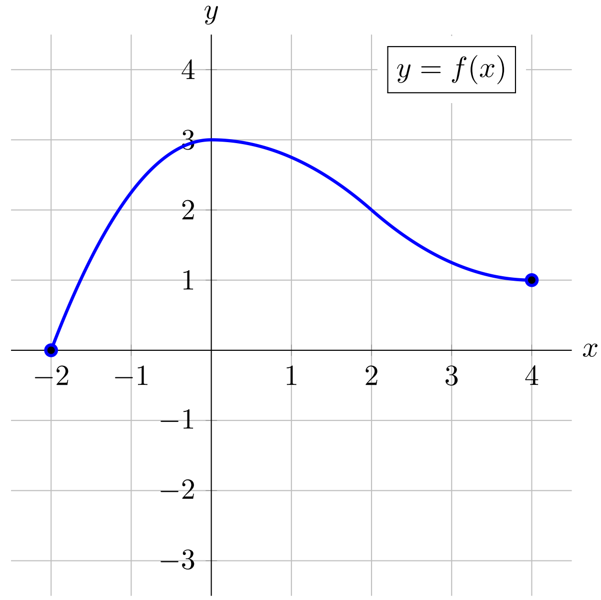
Answer: -36

2. [15 points]

a. [6 points] Carefully draw the graph of a single function $y = f(x)$ on the given axes that satisfies all of the given conditions.

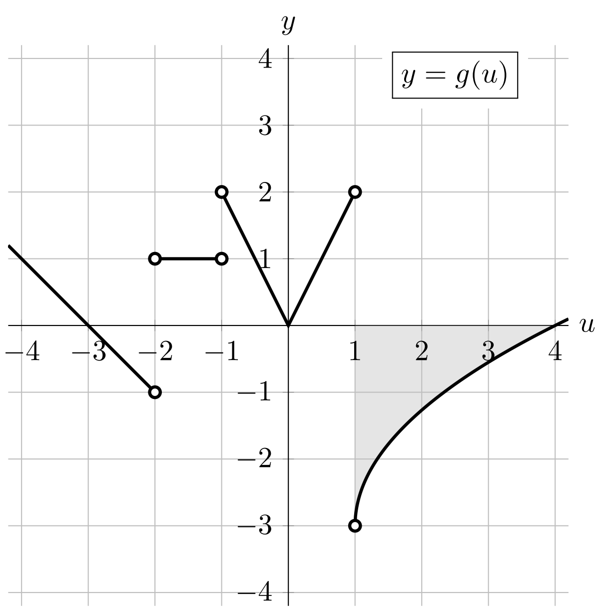
A **differentiable** function $f(x)$ with domain containing the interval $(-2, 4)$ such that:

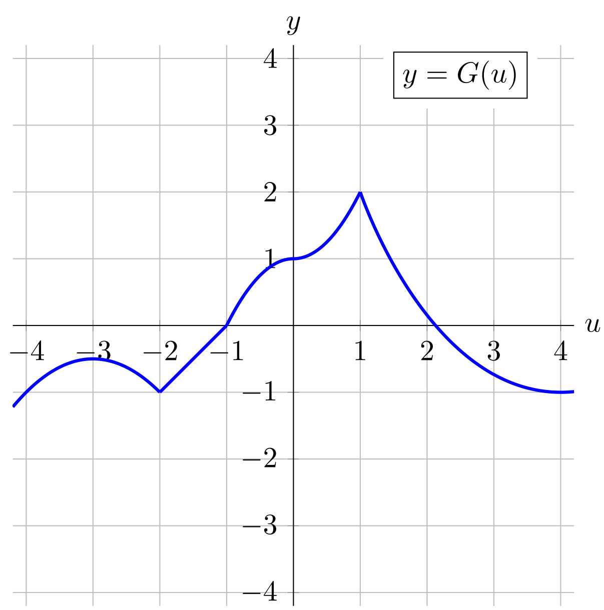
- $f(x)$ is increasing on the interval $(-2, 0)$ and decreasing on the interval $(0, 4)$;
- $f'(x)$ is decreasing on the interval $(-2, 2)$ and increasing on the interval $(2, 4)$;
- $\int_{-2}^{2} f'(t)\,dt = 2$.



b. [9 points] A portion of the graph of the function $g(u)$ is shown below on the left. **Carefully sketch** a continuous antiderivative $G(u)$ of $g(u)$ for $-4 < u < 4$ on the given axes on the right such that $G(0) = 1$.

- **Label the points** (u, y) on your sketch of $G(u)$ with the correct y-value at the u-values $u = -4, -3, -2, -1, 0, 1, 4$.
- Note that $g(u)$ is linear on the intervals $(-4, -2)$, $(-2, -1)$, $(-1, 0)$, and $(0, 1)$, and that the shaded region has area 3.

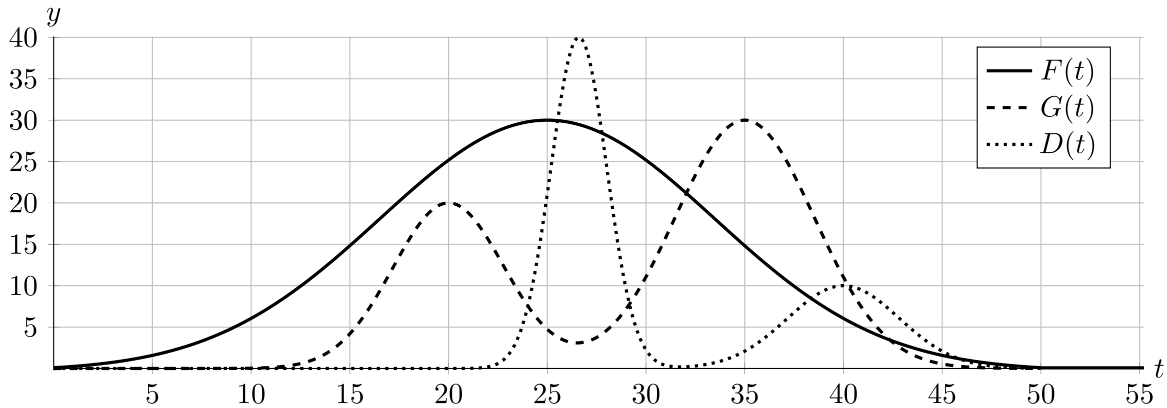




3. [10 points] A lone oak tree near the boundary between the Graham and Duckworth estates is shedding its leaves throughout the autumn season. Assume all the oak tree's leaves fall onto the Graham estate, at a rate of $F(t)$ thousand leaves per day, t days after 12am on Sept 30$^{\text{th}}$.

Graham, feuding with Duckworth, uses a leaf-blower to blow the oak tree leaves onto Duckworth's estate at a rate of $G(t)$ thousand leaves per day, t days after 12am on Sept 30$^{\text{th}}$. Not to be outdone, Duckworth blows the oak tree leaves back onto Graham's estate at a rate of $D(t)$ thousand leaves per day, t days after 12am on Sept 30$^{\text{th}}$. Throughout this problem, assume that all oak tree leaves on either estate have originally fallen from this tree.

Graphs of the continuous functions $F(t)$, $G(t)$, and $D(t)$ are shown below. Note that, for instance, $t = 15$ corresponds to 12am on Oct 15$^{\text{th}}$, and $t = 35$ corresponds to 12am on Nov 4$^{\text{th}}$.



a. [2 points] Approximately how many leaves fell from the tree onto Graham's estate between 12am Oct 20$^{\text{th}}$ and 12am Oct 25$^{\text{th}}$? Give a numerical answer, rounded to the nearest 10,000.

Answer: 140,000 leaves.

b. [2 points] Write an expression involving one or more integrals for the total number of oak tree leaves that are on Graham's estate 55 days after 12am on Sept 30$^{\text{th}}$.

Answer: $1000 \int_0^{55} \big(F(t) - G(t) + D(t)\big)\, dt$

c. [2 points] Estimate the rate of change, in thousands of leaves per day, of oak tree leaves on Graham's estate at 12am on Nov 4$^{\text{th}}$.

Answer: -13 **(or** $-13{,}000$ **leaves per day)**

d. [2 points] On which single calendar day was the *increase* in the number of oak tree leaves on Graham's estate the greatest? Give your best estimate.

Answer: **October 26th**

e. [2 points] On which single calendar day was the total *number* of oak tree leaves on Graham's estate the greatest? Give your best estimate.

Answer: **November 1$^{\text{st}}$**

4. [4 points] Evaluate the following limits. No justification necessary. Write DNE if a limit does not exist or diverges to $\pm\infty$.

i. $\displaystyle\lim_{x\to\infty} \frac{(3x^2+1)(x^3-2x+5)}{1-x-x^2-x^3-x^4}$

Answer: **DNE**

ii. $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

Answer: 0

iii. $\displaystyle\lim_{x\to\infty} \left(\frac{xe^{-x}+e^x}{xe^{-x}-e^x}\right)$

Answer: -1

iv. $\displaystyle\lim_{x\to\infty} \frac{(3+x^{-1})^2-9}{x^{-1}}$

Solution: If we let $f(x) = x^2$, then this limit is $f'(3)$, which is 6. Alternatively,

$$\lim_{x\to\infty} \frac{(3+x^{-1})^2-9}{x^{-1}} = \lim_{x\to\infty} \frac{9+6x^{-1}+x^{-2}-9}{x^{-1}} = \lim_{x\to\infty} \left(6+x^{-1}\right) = 6.$$

Answer: 6

5. [6 points] A portion of the graph of the differentiable function $f(x)$ is shown below to the right. Using this graph, put the eight quantities (a) through (h) *in order from least to greatest* by writing the letters on the blanks. You are given that $f(x)$ is quadratic on the interval $(2, 4)$.

(a) $f'(0)$

(b) $f'(0.5)$

(c) $f'(2)$

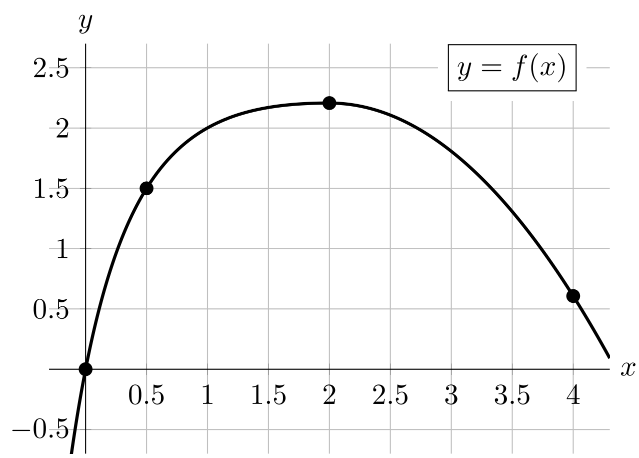
(d) $f''(3)$

(e) the average rate of change of $f(x)$ over $[0, 0.5]$

(f) the average rate of change of $f(x)$ over $[0.5, 2]$

(g) the average rate of change of $f(x)$ over $[0, 4]$

(h) the number 1



Answer: d , c , g , f , h , b , e , a

LEAST GREATEST

6. [10 points] Let $g(x)$ be the function defined by

$$g(x) \;=\; x\sin(\pi x) + \frac{1}{\pi}\cos(\pi x).$$

The derivative of $g(x)$ is

$$g'(x) \;=\; \pi x\cos(\pi x).$$

As a reminder, $1 = \cos(0) = \sin(\frac{\pi}{2})$, and $-1 = \sin(-\frac{\pi}{2}) = \cos(\pi) = \cos(-\pi)$.

a. [3 points] Find all critical points of the function $g(x)$ that are in the interval $[-\frac{1}{2}, 1]$.

Solution: We need to find points in $[-\frac{1}{2}, 1]$ such that

$$g'(x) = \;\pi x\cos(\pi x) = 0.$$

We get $x = 0$ or $\cos(\pi x) = 0$.
Next, $\cos^2(\pi x) = 1 - \sin^2(\pi x) = 0$, so $\sin(\pi x) = \pm 1$. Now, from $\sin(-\frac{\pi}{2}) = -1$ and $\sin(\frac{\pi}{2}) = 1$ it follows that $x = \pm 1/2$.

Answer: $x =$ $-1/2$, 0, $1/2$

b. [5 points] Find all x-values where the global extrema of $g(x)$ occur on the interval $[-\frac{1}{2}, 1]$. Be sure to show your work and justify your answers.

Solution: Let us compute the values of g at the critical points and end points:

$$\begin{aligned} g(-1/2) &= -\frac{1}{2}\sin(-\pi/2) + \frac{1}{\pi}\cos(-\pi/2) = 1/2,\\ g(0) &= \frac{1}{\pi}\cos(0) = 1/\pi,\\ g(1/2) &= \frac{1}{2}\sin(\pi/2) + \frac{1}{\pi}\cos(\pi/2) = 1/2,\\ g(1) &= \sin(\pi) + \frac{1}{\pi}\cos(\pi) = -1/\pi. \end{aligned}$$

We note that $\pi > 2$, so $1/\pi < 1/2$.

Answer: The maximum occurs at $x =$ $-1/2$, $1/2$

Answer: The minimum occurs at $x =$ 1

c. [2 points] Find a formula for the linear approximation $L(x)$ of the function $g(x)$ at the point $\big(-2, \frac{1}{\pi}\big)$.

Solution:

$$L(x) \;=\; g(-2) + g'(-2)\cdot(x+2) \;=\; \frac{1}{\pi} - 2\pi\cos(-2\pi)\cdot(x+2) \;=\; \frac{1}{\pi} - 2\pi(x+2)$$

Answer: $L(x) =$ $1/\pi - 2\pi(x+2)$

7. [8 points] Suppose $p(t) = t^{\sqrt{1/(t+1)}}$ gives the position of an object moving along a straight line, in meters east of a fixed starting point, t seconds after it begins moving. Note that the derivative $p'(t)$ outputs both positive and negative values for t-values in the interval $[0, 20]$. Match each expression on the right below with the letter (a) – (e) that it represents, or else write (f) if it does not represent any of (a) – (e). *Note: each letter (a) – (f) may appear more than once, or not at all.*

(a) the object's instantaneous velocity at $t = 20$

(b) the object's average velocity over the time interval $[0, 20]$

(c) the amount of time it takes for the object to travel 20 meters

(d) the total distance the object traveled over the time interval $[0, 20]$

(e) the distance between the object's location at $t = 0$ and its location at $t = 20$

(f) none of (a) – (e)

i. (b) $\dfrac{20^{\sqrt{1/21}} - 0^{\sqrt{1/1}}}{20 - 0}$

ii. (f) $\displaystyle\int_0^{20} t^{\sqrt{1/(t+1)}}\, dt$

iii. (f) $\dfrac{(20+h)^{\sqrt{1/(21+h)}} - 20^{\sqrt{1/21}}}{h}$

iv. (e) $\displaystyle\int_0^{20} p'(t)\, dt$

v. (d) $\displaystyle\int_0^{20} |p'(t)|\, dt$

vi. (a) $\displaystyle\lim_{h\to 0} \frac{(20+h)^{\sqrt{1/(21+h)}} - 20^{\sqrt{1/21}}}{h}$

vii. (f) $\dfrac{t^{\sqrt{1/21}} - 20^{\sqrt{1/21}}}{t - 20}$

viii. (e) $20^{\sqrt{1/21}} - 0^{\sqrt{1/1}}$

8. [9 points] Consider the family of functions given by

$$P(t) = \frac{A}{1 + Be^{-kt}}$$

where A, B, and k are **positive** constants. The first and second derivatives of $P(t)$ are

$$P'(t) = \frac{ABke^{kt}}{(B + e^{kt})^2} \qquad \text{and} \qquad P''(t) = \frac{ABk^2e^{kt}(B - e^{kt})}{(B + e^{kt})^3}.$$

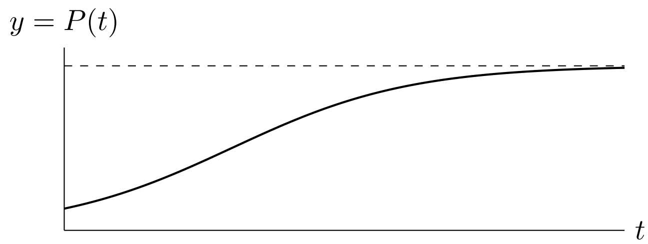
a. [3 points] Find all zeros of $P'(t)$ and $P''(t)$. Your answers may involve the constants A, B, and k. If there are none of a particular type, write NONE. *Hint: Remember that A, B, and k are just positive constants.*

Solution: $P'(t) = 0$ when $ABke^{kt} = 0$, but $e^x > 0$ for all inputs x, and we are given that A, B, and k are positive, so $P'(t)$ has no zeros. By similar reasoning, $P''(t) = 0$ when $B = e^{kt}$, that is, when $t = \frac{\ln B}{k}$.

Answer: $P'(t)$ has zero(s) at $t =$ **NONE**

$P''(t)$ has zero(s) at $t =$ $\frac{\ln B}{k}$

Researchers have demonstrated that, for appropriate values of A, B, and k, the function $P(t)$ is a good model for the total amount of oil produced in the US over the t years since 1950, in billions of barrels. For these particular values, a graph of $P(t)$ for $t \geq 0$ is shown below, where $t = 0$ corresponds to the start of 1950.



It is known or estimated that

- $\lim_{t\to\infty} P(t) = 180$, that is, US oil reserves would be depleted after using 180 billion barrels,
- at the start of 1950, a total of 40 billion barrels of oil had been produced in the US, and
- $P(t)$ was increasing the fastest, that is, the rate of oil production was largest, in 1970.

b. [6 points] Find the exact values of A, B, and k for this model. You do not need to simplify.

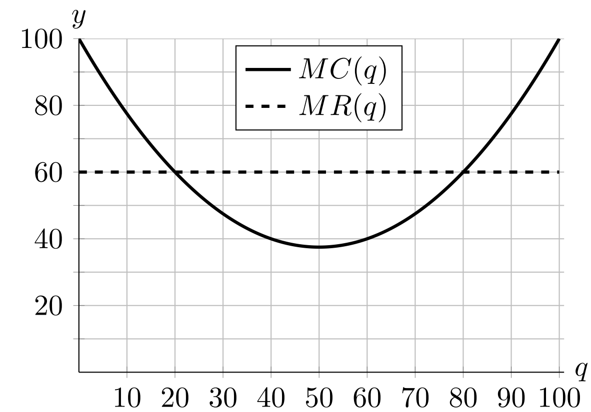
Solution: Since $\lim_{t\to\infty} Be^{-kt} = 0$, we know $\lim_{t\to\infty} P(t) = A$, so $A = 180$. The second bullet gives us that $40 = P(0) = \frac{A}{1+B}$, which we can solve for B to obtain $B = \frac{A}{40} - 1 = \frac{7}{2}$. Finally, $P(t)$ increases fastest at the inflection point in its graph, where $P''(t) = 0$, which happens at $t = \frac{\ln B}{k}$ by part a. Thus $20 = \frac{lnB}{k}$, which means $k = \frac{\ln B}{20} = \frac{\ln(7/2)}{20}$.

Answer: $A =$ 180 $\quad B =$ $\frac{7}{2}$ $\quad k =$ $\frac{\ln(7/2)}{20}$

9. [10 points] Olivia knits and sells custom-made clothing online, and she is considering adding hats, scarves, and mittens to her inventory this winter. She has modeled the expected cost and revenue for each of these items, and wants to know how many of each she should produce in order to maximize profit. In each part below, help her decide based on the given information.

a. [2 points]

Olivia can make at most 100 hats, and she has calculated the marginal cost $MC(q)$ and marginal revenue $MR(q)$ of producing and selling q hats, in dollars per hat, as shown in the graphs on the right.

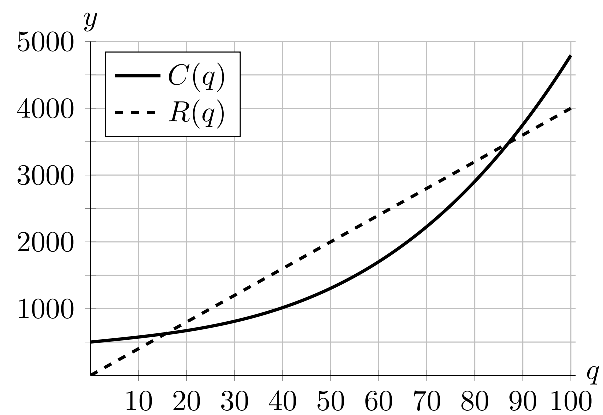


How many hats should she make in order to maximize profit? *Circle the one correct answer below.*

0 20 50 $\boxed{80}$ 100 NONE OF THESE

b. [2 points]

Olivia can make at most 100 scarves, and she has calculated the cost $C(q)$ and revenue $R(q)$ of producing and selling q scarves, in dollars, as shown in the graphs on the right.



Approximately how many scarves should she make in order to maximize profit? *Circle the one correct answer below.*

0 17 $\boxed{55}$ 87 100 NONE OF THESE

c. [6 points] Olivia can make at most 300 pairs of mittens, and she has calculated the cost $C(q)$ and revenue $R(q)$ of producing and selling q hundred pairs of mittens, in hundreds of dollars, to be

$$R(q) = 30q \qquad \text{and} \qquad C(q) = 2q^3 - 9q^2 + 42q.$$

How many hundreds of pairs of mittens should Olivia make in order to maximize profit? *Show all your work and use calculus to find your answer.*

Solution: Profit, $\pi(q)$, is equal to revenue minus cost, so

$$\pi(q) \;=\; R(q) - C(q) \;=\; 30q - (2q^3 - 9q^2 + 42q) \;=\; -2q^3 + 9q^2 - 12q.$$

In order to maximize $\pi(q)$ on the interval $[0, 300]$, we differentiate $\pi(q)$ to find its critical points:

$$\pi'(q) \;=\; -6q^2 + 18q - 12 \;=\; -6(q^2 - 3q + 2) \;=\; -6(q-1)(q-2).$$

So the critical points of $\pi(q)$ are $q = 1$ and $q = 2$, and now we just calculate $\pi(q)$ at these critical points and at the endpoints of our interval:

$$\pi(0) = 0, \qquad \pi(1) = -5, \qquad \pi(2) = -4, \qquad \pi(3) = -9.$$

This means Olivia should not make any mittens!

Answer: 0 hundreds of pairs of mittens

10. [8 points]

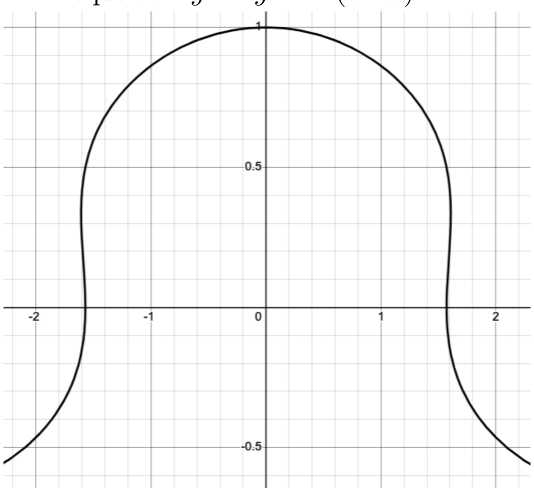
a. [5 points] The equation $y^3 + x^3 = y\cos x$ defines y implicitly as a function of x. Compute $\dfrac{dy}{dx}$. Show every step of your work.

Solution: Implicitly differentiating the equation $y^3 + x^3 = y\cos x$ with respect to x, we get

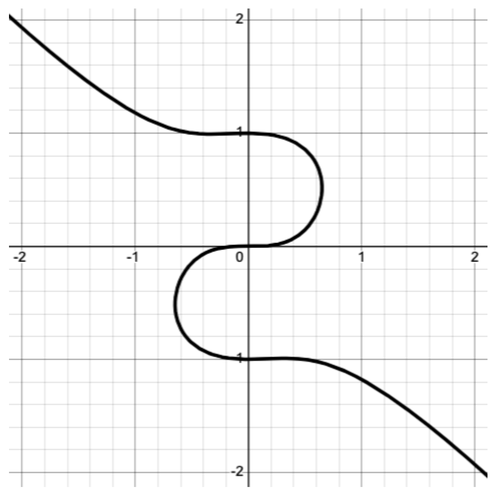
$$\begin{aligned} 3y^2\frac{dy}{dx} + 3x^2 &= \frac{dy}{dx}\cos x - y\sin x \\ \left(3y^2 - \cos x\right)\frac{dy}{dx} &= -3x^2 - y\sin x \\ \frac{dy}{dx} &= \frac{-3x^2 - y\sin x}{3y^2 - \cos x}. \end{aligned}$$

Answer: $\dfrac{dy}{dx} = \underline{\dfrac{-3x^2 - y\sin x}{3y^2 - \cos x}}$

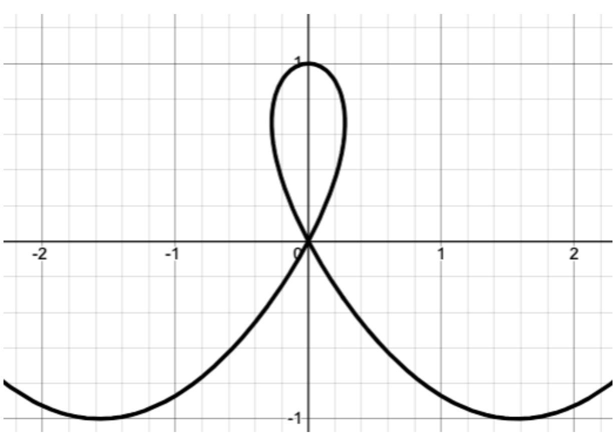
b. [3 points] Circle the number corresponding to the *one* graph below that is the graph of the equation $y^2 - y^3 = 2(\sin x)^2$.



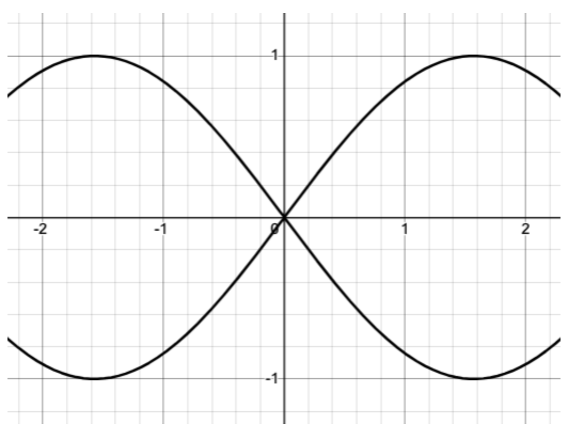
(i)



(ii)

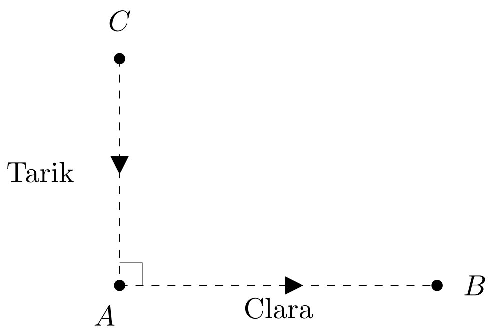


$\boxed{\text{(iii)}}$



(iv)

11. [7 points] Two birds, Tarik and Clara, are collecting materials for their nest. Clara is flying straight from the nest at point A to the pond at point B, and Tarik is flying straight from the forest at point C toward the nest at point A. At the moment shown in the diagram below, Tarik spots Clara flying toward the pond, and Clara is 40 meters from the nest, while Tarik is 30 meters from the nest. Note that each bird is represented in the figure by a triangle, pointing in the direction they are flying.



a. [2 points] How far apart are Tarik and Clara at the moment he spots her? **Include units.**

Solution: By the Pythagorean Theorem, the distance between Tarik and Clara at the moment he spots her is

$$\sqrt{30^2+40^2} = \sqrt{900+1600} = \sqrt{2500} = 50 \text{ meters.}$$

Answer: **50 meters**

b. [5 points] If Tarik's speed is 6 meters per second and Clara's is 4 meters per second at the moment he spots her, how fast is the distance between them changing at this moment? Give your answer exactly, and **include units.** Are they getting closer to each other or farther away from each other at this moment? Circle your answer below. You must justify your answer through your work to receive credit for the answer you circle.

Solution: We view the dashed lines in the figure above as coordinate axes with point A as the origin, and let x represent Clara's position between A and B, and y Tarik's position between A and C. So at the moment Tarik spots Clara, we have $x = 40$, $y = 30$, $\frac{dx}{dt} = 4$, and $\frac{dy}{dt} = -6$. Letting z be the distance between Tarik and Clara, we have that x, y, and z all depend on t and satisfy $x^2 + y^2 = z^2$ by the Pythagorean Theorem. differentiating this equation with respect to t gives us

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Plugging in $x = 40$, $y = 30$, $\frac{dx}{dt} = 4$, and $\frac{dy}{dt} = -6$, along with $z = 50$ from part a., we get

$$100\frac{dz}{dt} = 2(40)(4) + 2(30)(-6),$$

and solving this for $\frac{dz}{dt}$ gives

$$\frac{dz}{dt} = \frac{320-360}{100} = -\frac{2}{5}.$$

This means that, at the given moment, the distance between the two birds is *decreasing* at a rate of $\frac{2}{5}$ meters per second.

Answer: $-2/5$ **meters per second**

The birds are getting: $\boxed{\textbf{closer to each other}}$ **farther away from each other**