Math 115 — Midterm Exam — October 21, 2025

EXAM SOLUTIONS

1. Please neatly write your 8-digit UMID number, your initials, your instructor's first and/or last name, and your section number in the spaces provided.
2. This exam has 10 pages including this cover.
3. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it.
6. If you need more space to answer a question, please use the back of an exam page. Clearly indicate on your exam if you are using the back of a page, and also clearly label the problem number and part you are doing on the back of the page.
7. You are allowed notes written on two sides of a $3'' \times 5''$ note card. You are NOT allowed other resources, including, but not limited to, notes, calculators or other electronic devices.
8. For any graph or table that you use to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how you used the graph or table to find the answer.
9. Include units in your answer where that is appropriate.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	7	
2	9	
3	8	
4	5	

Problem	Points	Score
5	12	
6	9	
7	10	
Total	60	

1. [7 points] Let

$$p(w) = \frac{w}{1+e^w}.$$

a. [2 points] Evaluate each of the limits below. If a limit does not exist, including if it diverges to $\pm\infty$, write DNE. You do not need to show work.

i. $\lim_{w\to\infty} p(w)$

Answer: 0

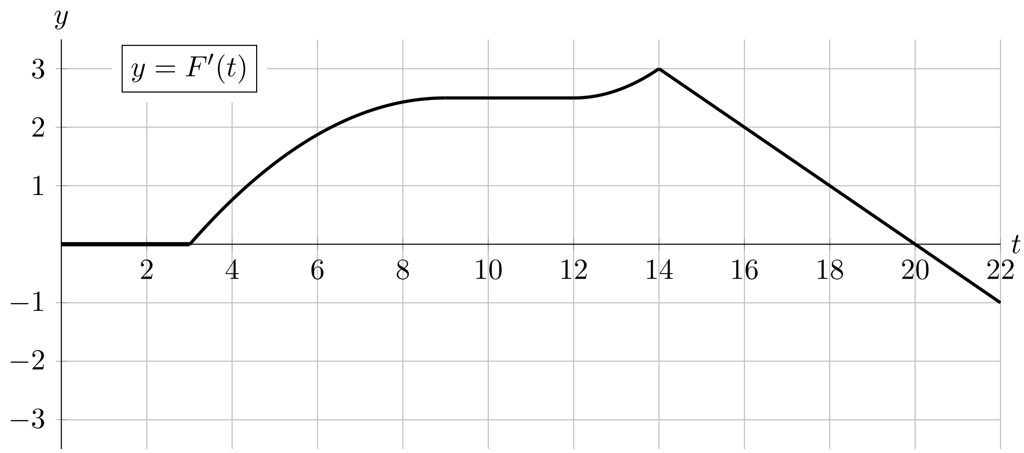
ii. $\lim_{w\to-\infty} p(w)$

Answer: **DNE** $(-\infty)$

b. [5 points] Use the limit definition of the derivative to write an explicit expression for $p'(2)$. *Your answer should not involve the letter p. Do not attempt to evaluate or simplify the limit.* Write your final answer in the answer box provided below.

Answer: $p'(2) = \lim_{h\to 0} \dfrac{\dfrac{2+h}{1+e^{2+h}} - \dfrac{2}{1+e^2}}{h}$

2. [9 points] An environmentalist for a local county has found some data on the county's forest cover. In particular, letting the "forest cover", $F(t)$, be the percentage of the county (by area) that was covered by forest t years after the start of 2000, the environmentalist has the graph of $F'(t)$, the **derivative** of $F(t)$, as shown below.



a. [3 points] At the start of 2016, was the forest cover increasing or decreasing? Circle your answer below. At approximately what rate? *Include units.*

INCREASING   DECREASING   at a rate of ≈ **2% per year**

b. [2 points] During which of the following years, if any, did the amount of forest cover appear to remain constant for the entire year? Circle **all** correct answers.

2001   2010   2018   2020   NONE OF THESE

c. [1 point] Which **one** of the following statements **best** describes the forest cover during 2004? Circle the numeral of your answer.

i. The amount of forest cover increased faster and faster as the year went on.

ii. The amount of forest cover was increasing, but grew more slowly as the year went on.

iii. The amount of forest cover increased at a constant rate.

d. [3 points] At which of the following times t did the county have the most forest cover? Circle the **one** correct answer, then **explain** in *two sentences or fewer* how you know.

$t = 0$ $t = 3$ $t = 14$ $t = 20$ $t = 22$

Explanation:

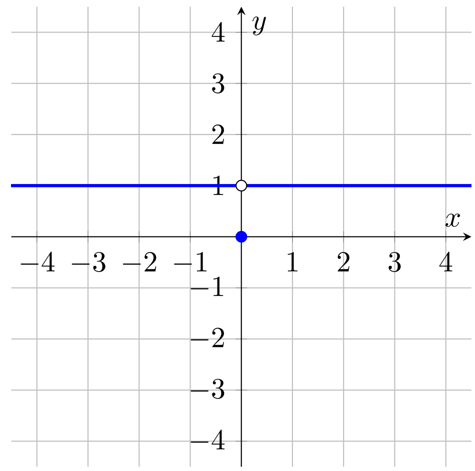
Solution: Since this is the graph of F', when the graph is positive, the amount of forest cover is growing. It therefore grows signficantly from 2003 until 2020 (after being constant from 2000-2003), and then only decreases by a small amount from 2020-2022, so 2020 must be when the forest cover was greatest.

3. [8 points] For each part below, using the axes to the right, carefully draw the graph of a single function with domain $[-4,4]$ that satisfies the given conditions. If it is not possible to do so, write NOT POSSIBLE and briefly explain why.

Make sure that any graph you draw is clear and unambiguous, and that you have carefully plotted any important points.

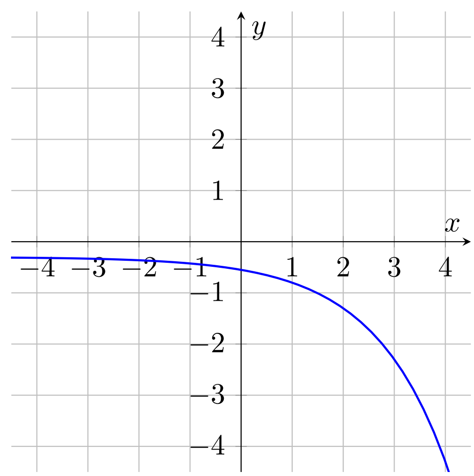
a. [2 points]

A function f with $f(0) = 0$ and $\lim_{x \to 0} f(x) = 1$.



b. [3 points]

A function g so that $g(x)$, $g'(x)$, and $g''(x)$ are each negative for all $-4 < x < 4$.



c. [3 points]

A function k with the following properties:

- $k(-3) = k(3) = 2$.
- $k(x)$ is differentiable on $(-4, 4)$.
- $k'(x)$ is never 0.

Solution: NOT POSSIBLE. k is differentiable and continuous on $[-3,3]$. By the Mean Value Theorem, there must exist a point c in the interval $[-3,3]$ such that

$$k'(c) = \frac{k(3) - k(-3)}{3 - (-3)} = \frac{2-2}{6} = 0.$$

4. [5 points] For each part of this problem, circle the word that can be inserted in the blank to correctly complete the sentence.
You do not need to show any work for this problem.

a. [1 point] If $P(x)$ and $Q(x)$ are two exponential functions, then $P(x)$ can ______________ be written as a transformation of $Q(x)$.

ALWAYS   SOMETIMES   NEVER

b. [1 point] If $a(x)$ and $b(x)$ are defined on all real numbers x, and $b(x)$ is not continuous, then $a(b(x))$ is ______________ continuous.

ALWAYS   SOMETIMES   NEVER

c. [1 point] If $\ell(x)$ and $m(x)$ are continuous at all x-values in $[-10, 10]$ with $m(2) = -1$ and $m(4) = 3$, then $n(x) = \frac{\ell(x)}{m(x)}$ is ______________ continuous at all x-values in $[-10, 10]$.

ALWAYS   SOMETIMES   NEVER

d. [1 point] If a is a positive real number, then the equation $1 = \ln(a - x^2)$ ______________ has at least one solution for x.

ALWAYS   SOMETIMES   NEVER

e. [1 point] If $C(t)$ is a sinusoidal function with a global minimum at $t = 1$, then $C(1.5)$ is ______________ equal to $C(0.5)$.

ALWAYS   SOMETIMES   NEVER

5. [12 points] Abby is trying to finish writing an essay at a cafe while drinking coffee continuously. At this cafe, Abby only drinks the coffee that costs \$3 per cup, and the cafe does not sell fractional cups. Let $T(z)$ be the number of hours Abby has worked when she has consumed z milligrams of caffeine. Let $C(z)$ be the number of cups of coffee Abby needs to purchase to consume z milligrams of caffeine. Suppose T and T^{-1} are each both invertible and differentiable.

a. [2 points] The function C is not invertible. Explain why, using two sentences or fewer.

Solution: If a cup of coffee has 100 milligrams of caffeine, the we have $C(1) = C(2)$. Thus, C must not be invertible.

b. [3 points] Write a single **equation** involving T, T^{-1} and/or C that represents the following statement:

Abby has spent \$15 on coffee after working at the cafe for 4 hours.

Answer: $3C(T^{-1}(4)) = 15$

c. [3 points] Write a single **equation** involving T', $(T^{-1})'$ and/or C' that represents the following statement:

After 0.05 hours of working on her essay at the cafe, Abby has consumed approximately 1 milligram of caffeine.

Answer: $(T^{-1})'(0.05) = 20$, $(T^{-1})'(0) = 20$, $T'(0) = 0.05$, **or** $T'(1) = 0.05$

5. (continued) The setup of the problem is restated here for your convenience.

Abby is trying to finish writing an essay at a cafe while drinking coffee continuously. At this cafe, Abby only drinks the coffee that costs \$3 per cup, and the cafe does not sell fractional cups. Let $T(z)$ be the number of hours Abby has worked when she has consumed z milligrams of caffeine. Let $C(z)$ be the number of cups of coffee Abby needs to purchase to consume z milligrams of caffeine. Suppose T and T^{-1} are each both invertible and differentiable.

d. [2 points] Suppose $T(95) = 3/2$ and $T'(95) = 1/60$. Give a formula for $L(z)$, the local linearization of T near $z = 95$.

Answer: $L(z) =$ $\frac{1}{60}(z-95)+\frac{3}{2}$

e. [2 points] Use your linear approximation from the previous part to estimate how many milligrams of caffeine Abby has consumed after 1 hour and 36 minutes. Note that there are 60 minutes in one hour. You must show work supporting your final answer.

Solution: We set $L(z) = 1.6$ Then,

$$\frac{1}{60}(z-95)+1.5 = 1.6$$

$$\frac{1}{60}(z-95) = .1$$

$$z - 95 = 6$$

$$z = 101.$$

Answer: 101 milligrams

6. [9 points] In this problem, we consider the function $f(x) = 4xe^{-x^2/8}$. In case it is helpful, recall that $e \approx 2.7$. The first derivative of $f(x)$ is given by:

$$f'(x) = (4 - x^2)e^{-x^2/8}.$$

For each part of this problem, **be sure you show enough evidence** to support your conclusions.

a. [5 points] Find the x-coordinates of all critical points of $f(x)$ on $(-\infty, \infty)$. Find the x-coordinates of the local maxima and local minima of $f(x)$, or write NONE in the appropriate blank if there are none of the specified type.

Solution: The critical points of $f(x)$ are the values where $f'(x)$ is zero or undefined. Here, $f'(x)$ is always defined. $f'(x) = 0$ when $(4 - x)^2 = (2 - x)(2 + x) = 0$. The critical points are $x = -2$ and $x = 2$.

To find the local max(es) and min(s), we apply the first derivative test to the critical points we found. We have $f'(x) = (2 - x)(2 + x)e^{-x^2/8}$. We then get the following sign chart:

$f'(x)$:	$(-\infty, -2)$	-2	$(-2, 2)$	2	$(2, \infty)$
	$+ \cdot - \cdot + = -$		$- \cdot - \cdot + = +$		$- \cdot + \cdot + = -$

From this, we see that $x = -2$ is a local minimum and $x = 2$ is a local maximum.

Answer: Critical point(s): $x =$ ± 2

Answer: Local Max(es): $x =$ 2 Local Min(s): $x =$ -2

b. [4 points] Find the global maximum and minimum of $f(x)$ on $[1, \infty)$, or write NONE if there is no global extremum of that type. Give both the x-value(s) where $f(x)$ achieves the global max/min and the value of $f(x)$ at that x-value.

Solution: By part (a), we have that $x = 2$ is the only critical point on the interval $[1, \infty)$. Since 2 is a local maximum, we have that $f(x)$ has a global maximumum of $4(2)e^{-\frac{2^2}{8}} = 8e^{-1/2}$. We have $f(1) = 4e^{-1/8} > 0$ and $\lim_{x\to\infty} 4xe^{-x^2/8} = 0$. Thus, $f(x)$ does not have a global minimum on this interval.

Answer: $f(x)$ has a global max value of $y =$ $8e^{-1/2}$ occurring at $x =$ 2

Answer: $f(x)$ has a global min value of $y =$ **NONE** occurring at $x =$ **NONE**

7. [10 points] Given below is a table of values for a **continuous** function $q(t)$ and its first and second derivative. The question marks indicate values that are not known. For each question mark, the value may or may not exist, but at all other real numbers t, the values of $q(t)$, $q'(t)$, and $q''(t)$ exist.

t	-5	-3	-1	1	3	5	7	9
$q(t)$	1	?	0	?	?	?	?	?
$q'(t)$	?	0	-3	0	1	2	?	-3
$q''(t)$	?	-5	0	2	0	1	?	2

Between any two consecutive values of t in the table, both $q'(t)$ and $q''(t)$ are either **always positive** or **always negative**.

a. [2 points] Estimate $q'''(-2)$. Show your work to justify your answer.

Solution:

$$q'''(-2) \approx \frac{q''(-1) - q''(-3)}{-1-(-3)} = \frac{0-(-5)}{2} = 2.5$$

Answer: $q'''(-2) \approx$ 2.5

b. [1 point] Is $t = -3$ a local minimum of $q(t)$, a local maximum of $q(t)$, or is there not enough information to decide (NEI)? Circle **one** answer only. (No justification needed.)

LOCAL MIN [LOCAL MAX] NEI

Solution: Because $q''(-3) < 0$, the second derivative test tells us that $t = -3$ must be a local maximum as q is concave down there.

c. [2 points] Which of the following values of t must be inflection points of $q(t)$? Circle **all** correct answers. (No justification needed.)

$t = -3$ [$t = -1$] $t = 1$ $t = 3$ NONE OF THESE

d. [2 points] Which of the following values must be undefined? Circle **all** correct answers. (No justification needed.)

$q'(-5)$ [$q'(7)$] $q''(-5)$ [$q''(7)$] NONE OF THESE

e. [3 points] Find the t-coordinate(s) of the **global minimum(s)** of $q(t)$ on the interval $[-5, 5]$, or write NEI if there is not enough information to determine this. Then, **briefly justify** your answer. You may give a possible sketch of $q(t)$ on $[-5, 5]$ as your justification.

Answer: Global min(s) at $t =$ 1

Justification:

Solution: The global minimum of $q(t)$ must be at $t = 1$. Because $t = -1$ and $t = 1$ are the only critical points of q inside this interval, and because the interval is closed and q is continuous (since it is differentiable), the global max(es) must occur at one of these points or an endpoint. We can see that $t = -5$ isn't a global min given the two known values of q. Also, $t = -3$ was a local max as determined above. Also, note that $q' > 0$ for $t > 1$, so q increases after this point. So the lowest value must occur at $t = 1$.