Math 115---Final Exam

Winter 2001

DEPARTMENT of MATHEMATICS
University of Michigan

April 20, 2001

Name: ______________________________ **Signature:** __________________________________

Instructor: __________________________________ **Section No:** ____________

General Instructions: *Do not open this exam until you are told to begin.* This test consists of 12 questions on 9 pages (including this cover sheet). The exam is worth 100 points. Do not separate the exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

Problem No.	Points	Score
1	5	
2	4	
3	3	
4	8	
5	3	
6	10	
7	18	
8	16	
9	3	
10	17	
11	3	
12	10	
Total	100	

(1.) (1 pt each) True / False--Circle your choice. Circle **T** only if the statement is ***always*** true. [No explanation necessary.]

(a) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$ for all x. **T** **F**

(b) If $f''(a) = 0$, then f has an inflection point at $x = a$. **T** **F**

(c) If $x = p$ is not a critical point of f, then $x = p$ is not a local maximum of f. **T** **F**

(d) If $\int_0^2 f(x)dx = 6$ then $\int_0^4 f(x)dx = 12$. **T** **F**

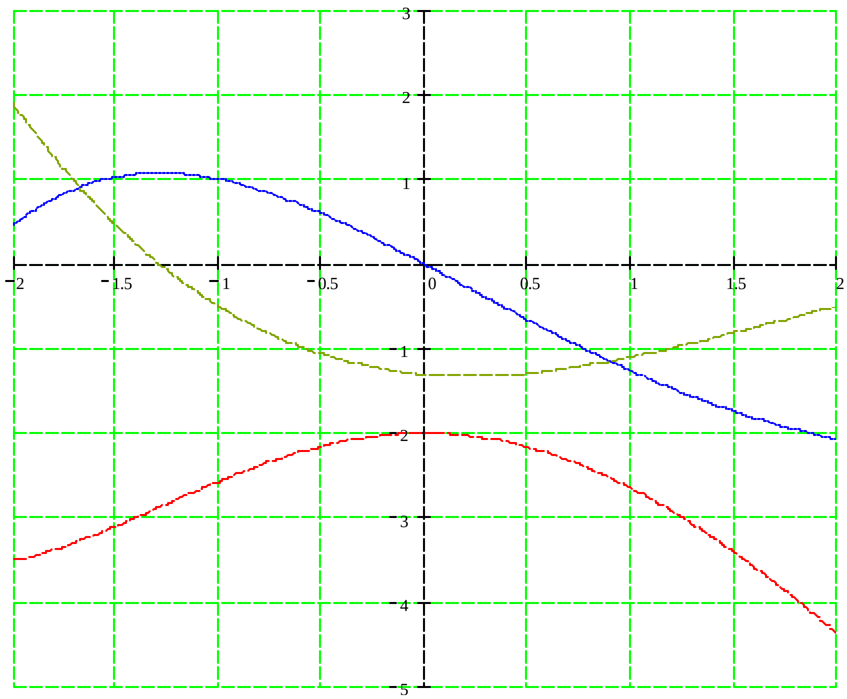
(e) If $\int_0^2 f(x)dx = 6$ and $h(x) = 5f(x)$ then $\int_0^2 h(t)dt = 30$. **T** **F**

(2.) (4 pts.) Is the function $g(x) = x^3 - \dfrac{x}{16}$ invertible? _______________

Below, give a clear justification for your answer.

(3.) (3 pts.) [No need to simplify, but show ***all*** of your work. **Circle** your answer.]
Find the derivative of $s(x) = \sin^5(3x^2 - 2)$.

(4.) **(8 pts.)** The graphs of f, f', and f'', are shown in the figure below. Determine which is which, and give ***clear* and *precise*** reasons for your choices.



[Note: On the original exam, these graphs were labeled.]

Graph A must be the graph of __________, because

Graph B must be the graph of ___________, because

Graph C must be the graph of ___________, because

(5.) (3pts) [Another derivative…. No need to simplify, but show ***all*** of your work, & **circle** your answer.]

Find the derivative of $r(w) = e^{5w}(w^2 - 4)^3$.

(6.) A gas leak is discovered in a large municipal building. The rate at which gas is leaking into the building is increasing, as indicated in the table below.

Time (hours)	0	7	14	21	28	35
Rate (grams/hour)	125	127	132	140	153	171

Initially there is no gas in the building. Suppose the building is well sealed so that no gas escapes from it.

(a) **(6 pts.)** Determine lower and upper estimates for the amount of gas in the building after 21 hours.

(b) **(4 pts.)** How often would the rate need to be measured over the interval from t = 0 to t = 35 in order to find upper and lower estimates within 100 grams of the actual amount of gas in the building?

(7.) The function $f(x) = x\ ln(x)$ has one critical point on the interval (0, 5).

(a) **(4 pts)** Determine the **exact** x value (i.e., not a decimal approximation) for the location of this critical point.

x = ____________

(b) **(3pts)** Is this point a maximum or a minimum or neither of these? Explain and show your work.

(c) **(2 pts)** Determine the instantaneous rate of change of f at $x = 1$ and at $x = 2$.

@ $x = 1$ ______________

@ $x = 2$ ______________

(d) **(2 pts)** What do the values in part (c) suggest about the concavity of the function between $x = 1$ and $x = 2$? Explain.

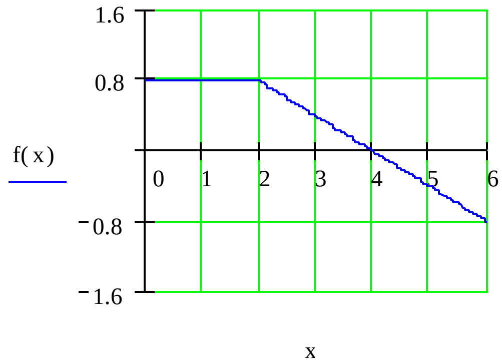
(e) **(3 pts)** Determine an equation of the tangent to the graph of f at $x = 1$.

____

(f) **(2 pts)** Use your equation from part (e) to approximate $f(2)$.

(g) **(2 pts)** Should your estimate be an underestimate or an overestimate?
_________________________ Why?

(8.) The graph in the figure below is the graph of $\frac{dh}{dt}$, where h is the altitude in thousands of feet above sea level and t is in hours, for Professor Bob's recent climb to the top of Bear Peak in Colorado. Use the graph to answer the following questions.



(a) (3 pts) How long did it take Bob to reach the peak of the mountain?

(b) (5 pts) What was the total change in altitude between $t = 0$ and $t = 4$?

(c) (4 pts) If Bob began his climb at 6000 feet above sea level, how high is the peak above sea level?

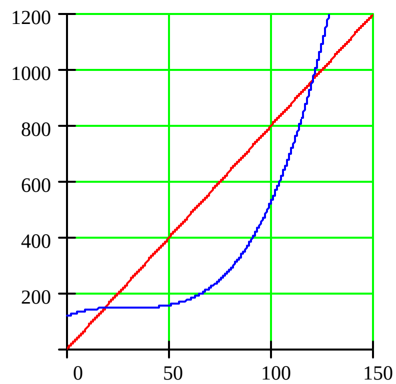
(d) (4pts) After 6 hours, Bob stopped at a lookout point to have a snack. What was the altitude of the lookout point?

(9.) (3 pts) Use the Fundamental Theorem of Calculus to evaluate the function below. To get credit, you must **show all of your work**. Please circle your answer.

[Note: This is a different problem from above.]

$$\int_2^5 (3x^2 - 4x + 1)dx$$

(10.) Cost and revenue functions for a charter bus company are shown in the figure below, where q is the number of buses that the company owns.



(a) **(4 pts)** Should the company add a 50^{th} bus? How about a 100^{th}? Explain your answers using marginal revenue and marginal cost. (You may illustrate your reasons graphically as well, if you like.)

(b) **(3 pts)** What does *C*¢ (50) = *A* (*A*, a constant) mean in the context of this problem? What are the units of the 50 and the units of *A*?

(c) **(4 pts)** Estimate the number of buses the company should have in order to maximize profit. Explain how you determined your estimate.

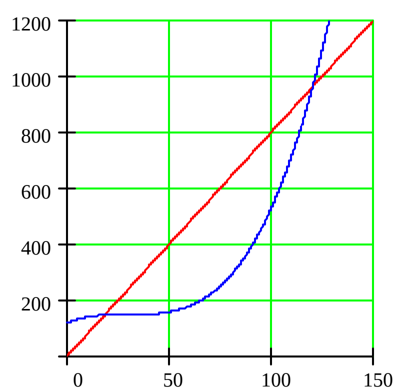
(Problem 10 continued)

(d) (6 pts)

(i) If the average cost, *a(q)*, is given by $a(q) = \dfrac{C(q)}{q}$, approximate q_0 so that $a(q_0)$ is the minimal average cost.

(ii) Show ***analytically*** that average cost will be minimized when $C'(q) = a(q)$.

(iii) Demonstrate on the graph below how this result can be shown graphically.



(11.) And, for good measure, one last derivative…. No need to simplify, but **show all your work.**

(3 pts) Find the derivative of $k(t) = \dfrac{(3t-4)}{\cos(2t)}$.

(12.) The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter.

(a) **(6 pts)** What choice of dimensions will make the rectangular area in the center as large as possible?

(b) **(4 pts)** What should the dimensions be if the total area enclosed by the running track is to be as large as possible?

Good luck, and a happy spring and summer to each of you!