(1.) (2 pts each) **True / False**--Circle your choice. Circle **T** only if the statement is ***always*** true. [No explanation necessary.]

(a) If a function is differentiable, then it is continuous. **T** **F**

(b) If a function is continuous, then it is differentiable. **T** **F**

(c) If $f'(x)$ is increasing, then f is concave up. **T** **F**

(d) If $f''(x) = -3$, then f is decreasing. **T** **F**

(e) If f has a critical point at $x=3$, then f has a local maximum or a local minimum at $x=3$. **T** **F**

(2.) Given :

$r(2) = 2$	and	$s(2) = 1$
$r(4) = -1$		$s(4) = 2$
$r'(2) = 5$		$s'(2) = 3$
$r'(4) = -3$		$s'(4) = 4$

Determine the values indicated below ***or*** state clearly what information is needed (and not supplied) to determine the requested value. In each case, first determine a general formula for the derivative function and then find the requested value (if possible). [Circle your answers.]

(3 pts each) Find:

(a) $H'(2)$ if $H(x) = \ln(r(x))$

$$H'(x) = \frac{1}{r(x)} \cdot r'(x) \rightarrow H'(2) = \frac{1}{r(2)} \cdot r'(2) = \frac{1}{2} \cdot 5 = \frac{5}{2}$$

(b) $H'(2)$ if $H(x) = \dfrac{s(x)}{r(x)}$

$$H'(x) = \frac{s'(x)r(x) - s(x) \cdot r'(x)}{(r(x))^2} \rightarrow H'(2) = \frac{3(2) - (1)(5)}{(2)^2}$$

$$= \frac{1}{4}$$

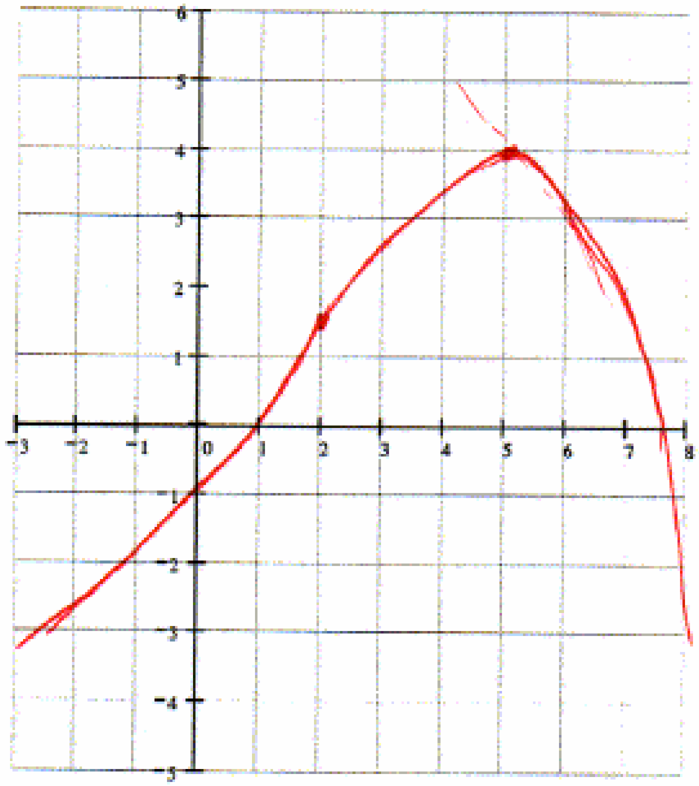
(c) $H'(2)$ if $H(x) = \sqrt{s(x)}$ $= (s(x))^{1/2}$

$$H'(x) = \frac{1}{2}(s(x))^{-1/2} \cdot s'(x)$$

$$\rightarrow H'(2) = \frac{1}{2}(s(2))^{-1/2} \cdot s'(2) = \frac{1}{2}(1)^{-1/2} \cdot 3 = \frac{3}{2}$$

(3.) (6 pts) (a) On the axes below, sketch a graph of a single *differentiable* function, $y = f(x)$, which has ***all*** of the following features:

- $f(5) = 4$ → $(5,4)$
- $f'(5) = -1$
- increasing: $f'(x) > 0$ for all $x < 4$
- conc. up $f''(x) > 0$ for all $x < 2$
- conc. down $f''(x) < 0$ for all $x > 2$
- decreasing: $f'(x) < 0$ for all $x > 4$



(b) (4 pts) Using the given information, find an equation of the line tangent to the graph of f at $x = 5$.

Given point $(5,4)$, slope $= -1$

$y - 4 = -1(x-5) = -x+5$

$y = -x + 9$

(c) (2 pts) Use your answer from part (b) to approximate $f(6)$.

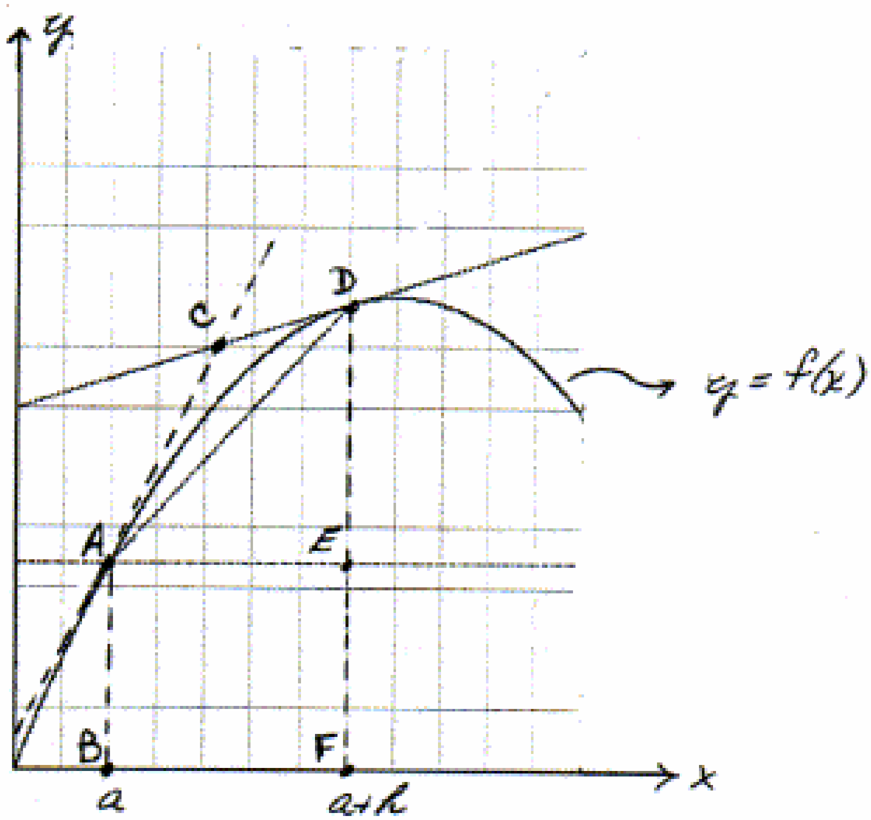
$f(6) \approx -6+9 = 3$

(d) (3 pts) From the ***given*** conditions (*i.e.*, not just from your graph), should the approximation in part (c) be an overestimate or an underestimate? Explain--using a complete sentence.

Since f is concave down at $x=6$, the approximation should be an overestimate.

(4.) Each of the expressions below describes either a ***length*** or a ***slope*** of one of the lines shown in the figure below. In the first blank following each expression, write either the word "length" or "slope," and in the second blank use the letters from the figure to identify the line or line segment. Each line or segment must be identified by **two** letters. For example, if you were given "h," the answer would be

h is the ____length____ of the line between ____B & F____ .



(10 pts—2 pts each)

(a) $f(a)$ is the ____length____ of the line between ____A & B____

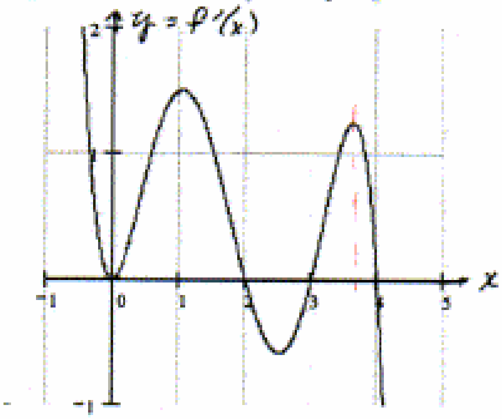
(b) $f(a+h)$ is the ____length____ of the line between ____D & F____

(c) $f(a+h)-f(a)$ is the ____length____ of the line between ____D & E____

(d) $\dfrac{f(a+h)-f(a)}{h}$ is the ____slope____ of the line between ____A & D____

(e) $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ is the ____slope____ of the line between ____A & C____

(5.) (14 pts) The graph in the figure below is the graph of $f'(x)$ (*i.e.*, the graph of the ***derivative*** of f). [Note: all questions refer to f, not f'.]



Graph of the derivative of f

(a) Determine *all* values of x for which:

(i) f has critical point(s)

$x = 0, 2, 3, 4$

(ii) f has local maximum(s)

$x = 2, \; x = 4$

(iii) f has local minimum(s)

$x = 3$

(iv) f has inflection point(s)

$x = 0, 1, 2.5, 3.7$

(b) Give one interval over which f is concave down.

[accept any interval over which f' is decreasing.]

eg: $1 < x < 2$
$1 < x < 2.5$
$-0.5 < x < 0$
$3.7 < x$
etc....

(c) Give the largest interval over which f is increasing.

$x < 2$ or $(-\infty, 2)$

Note: due to allowable interpretations, will accept:

$(-\infty, 2)$

$(-0.5, 2)$

or $(0, 2)$

[Note: (1,1) satisfies $1+2-3=0$ ✓]

(6.) (8 pts) [Show all work.]

If y satisfies the equation

$$y^2 + 2xy - 3x = 0,$$

(a) find $\frac{dy}{dx}$.

By implicit differentiation:

$$2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y - 3 = 0$$

so $\frac{dy}{dx}(2y+2x) = 3-2y$

$$\frac{dy}{dx} = \frac{3-2y}{2y+2x}$$

(b) Based on your answer to part (a), is the graph increasing, decreasing, or neither (*ie.*, tangent horizontal or undefined) at the point (1,1)? Explain.

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{3-2}{2+2} = \frac{1}{4}$$

Thus, the graph is increasing at the point (1,1) because the derivative (or slope) is positive at that point.

(7.) (12 pts) A laboratory study investigating the relationship between diet and weight in adult humans found that the weight, W, of a subject, in pounds, was a function, f, of the daily average number of calories, c, consumed by the subject. In terms of diet and weight, interpret the following statements or expressions. [Be certain to include units and write in sentences.]

Given: $W = f(c)$.

(a) $f(1800) = 155$ A person who consumes on average 1800 calories per day weighs 155 lbs.

(b) $f'(2000) = 0$ At 2000 daily average calories, the person's weight is stable -- neither increasing or decreasing. The person's weight will not change if they consume ~1 more calorie.

(c) $f^{-1}(162)$ The expression $f^{-1}(162)$ represents the average daily calories that a person weighing 162 lbs consumes.

Note: $f^{-1}(W) = c$

(d) What are the units of $f'(c)$?

The units of $f'(c)$ are in pounds per calorie.

(8.) (12 pts) From Exam I, we have that the population of Michigan can be approximated by

$$P = f(t) = 7.8(1.0058)^t,$$

where t is the number of years since the beginning of 1960 and P is in millions.

(a) Determine the average rate of change in the population of Michigan between 1960 and 1980. [Be certain to include units and express your answer as a complete sentence.]

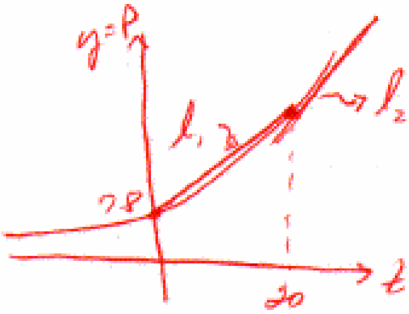
$$\frac{f(20)-f(0)}{20} = \frac{7.8(1.0058)^{20}-7.8}{20} \approx .04782$$

Over the 20-year period from 1960 to 1980, the population increased on average ~ 47,821 people per year.

(b) Determine the (instantaneous) rate of change of the population of Michigan at the beginning of 1980. [Again, use units and a sentence. Show your work.]

The instantaneous rate of change in 1980 is given by $f'(20) = 7.8(1.0058)^{20}\cdot\ln(1.0058) \approx$
Thus, in 1980, the population was increasing at the rate of $\approx .05064$ million people per year -- or ~ 50,640 people per year.

(c) Which is greater—the average rate of change between 1960 and 1980 or the instantaneous change in 1980? Use a graph or tables to give a convincing argument that the rate that you *found* to be greater should indeed *be* greater.



In the figure to the left, the slope of l_1 represents the average rate of change between 1960 + 1980. The slope of l_2 represents the instantaneous change in 1980. Clearly, the slope of l_2 is greater than the slope of l_1. Likewise, the inst. change is greater than the average between 1960 + 1980.

(d) Is there some time, t, such that the instantaneous rate of change of P is equal to the average rate of change from 1960 to 1980? If so, approximate t. If not, explain why not.

Yes, there will be a time between 1960 + 1980 when $f'(t) = \frac{f(20)-f(0)}{20}$.
We see this from the graph. Using a calculator to graph $y_1 = 7.8(\ln(1.0058))\cdot(1.0058)^x$ + $y_2 = .04782$, we find at $t \approx 10.126$ the slopes are equal.

(9.) (10 pts) Determine a, b, and c so that the graph of the function $f(x) = x^3 + ax^2 + bx + c$ has
a local maximum at $x = -2$,
a local minimum at $x = 1$,
and passes through the point (0,2). [Show your work.]

Given:
$f(x) = x^3 + ax^2 + bx + c$.

Since (0,2) is on the graph,
$f(0) = c = 2 \longrightarrow c = 2$

Also, since f is a polynomial, f' is defined for all x. Thus, to have max/min behavior, $f'(-2) = 0$ & $f'(1) = 0$.

Note: $f'(x) = 3x^2 + 2ax + b$

$f'(-2) = 3(4) + 2a(-2) + b = 0$

$\longrightarrow$ ① $12 - 4a + b = 0$

& $f'(1) = 3 + 2a + b = 0 \longrightarrow$ ② $3 + 2a + b = 0$

Subtracting ② from ① gives $9 - 6a = 0$,
so $9 = 6a \rightarrow a = \frac{3}{2}$

Substituting $a = \frac{3}{2}$ into ② gives

$3 + 2\left(\frac{3}{2}\right) + b = 0$,
$3 + 3 + b = 0$
$b = -6$

Thus, $f(x) = x^3 + \frac{3}{2}x^2 - 6x + 2$

$f(0) = 2$ ✓

Note:
$f''(x) = 6x + 3$
$f''(-2) = -12 + 3 = -9 < 0$
$\rightarrow$ loc. max!
$f''(1) = 6 + 3 = 9 > 0$
$\rightarrow$ loc. min!
by 2nd Deriv test.