(key)
Solution Guide

Math 115---Final Exam
Winter 2001

DEPARTMENT of MATHEMATICS
University of Michigan

April 20, 2001

Name: Key **Signature:** ______________________________

Instructor: ______________________________ **Section No:** ____________

General Instructions: *Do not open this exam until you are told to begin.* This test consists of 12 questions on 9 pages (including this cover sheet). The exam is worth 100 points. Do not separate the exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

Problem No.	Points	Score
1	5	
2	4	
3	3	
4	8	
5	3	
6	10	
7	18	
8	16	
9	3	
10	17	
11	3	
12	10	
Total	100	

(1.) (1 pt each) True / False--Circle your choice. Circle **T** only if the statement is *always* true. [No explanation necessary.]

(a) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$ for all x. T **F**

(b) If $f''(a) = 0$, then f has an inflection point at $x = a$. T **F**

(c) If $x = p$ is not a critical point of f, then $x = p$ is not a local maximum of f. T **F**

(d) If $\int_0^2 f(x)dx = 6$ then $\int_0^4 f(x)dx = 12$. T **F**

(e) If $\int_0^2 f(x)dx = 6$ and $h(x) = 5f(x)$ then $\int_0^2 h(t)dt = 30$. **T** F

(2.) (4 pts.) Is the function $g(x) = x^3 - \frac{x}{16}$ invertible? No

Below, give a clear justification for your answer.

NOTE: $g'(x) = 3x^2 - \frac{1}{16}$, so $g'(x) = 0$ if $x^2 = \frac{1}{48}$

$\rightarrow x = \pm\sqrt{\frac{1}{48}} \approx \pm .1443$

The function is increasing, then decreasing & then increasing. Since g is continuous, it is not invertible.

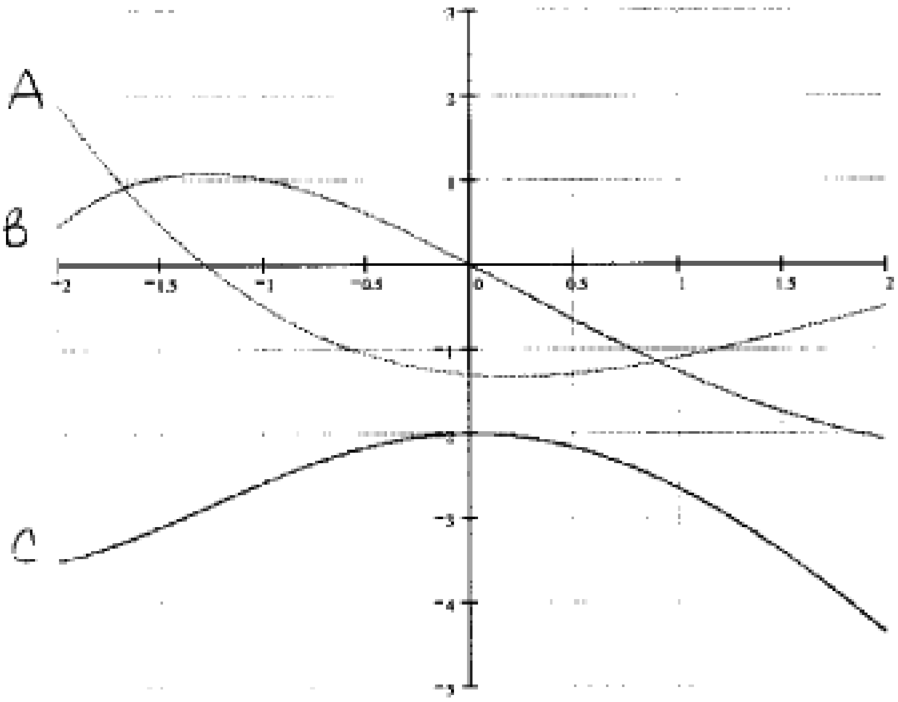
(3.) (3 pts.) [No need to simplify, but show *all* of your work. **Circle** your answer.]

Find the derivative of $s(x) = \sin^5(3x^2 - 2)$. $= [\sin(3x^2-2)]^5$

$s'(x) = 5\sin^4(3x^2-2) \cdot \cos(3x^2-2)(6x)$ ok to leave or

$s'(x) = 30x \sin^4(3x^2-2)\cos(3x^2-2)$

(4.) **(8 pts.)** The graphs of f, f', and f'', are shown in the figure below. Determine which is which, and give ***clear*** **and** ***precise*** reasons for your choices.



Graph A must be the graph of f'', because A is the graph of the derivative of B (f'). A is positive when B is increasing & negative when B is decreasing. Furthermore, A is positive when C is concave up & negative when C is concave down.

Graph B must be the graph of f', because B is positive where Graph C is increasing $(x<0)$ and negative where C is decreasing $(x>0)$.

Graph C must be the graph of f, because it cannot be the derivative of either of the other functions. Graph C is negative everywhere & neither A or B are decreasing throughout.

(5.) (3pts) [Another derivative.... No need to simplify, but show ***all*** of your work, & **circle** your answer.]

Find the derivative of $r(w) = e^{5w}(w^2-4)^3$.

[ok to leave.]

$$r'(w) = 5e^{5w}(w^2-4)^3 + 3e^{5w}(w^2-4)^2(2w)$$

or

$$= e^{5w}(w^2-4)^2\left[5(w^2-4)+6w\right]$$

$$= e^{5w}(w^2-4)^2\left[5w^2+6w-20\right]$$

(6.) A gas leak is discovered in a large municipal building. The rate at which gas is leaking into the building is increasing, as indicated in the table below.

Time (hours)	0	7	14	21	28	35
Rate (grams/hour)	125	127	132	140	153	171

Initially there is no gas in the building. Suppose the building is well sealed so that no gas escapes from it.

(a) (6 pts.) Determine lower and upper estimates for the amount of gas in the building after 21 hours.

Lower: $125(7) + 127(7) + 132(7) =$ 2688 gms

Upper: $127(7) + 132(7) + 140(7) =$ 2793 gms

(b) (4 pts.) How often would the rate need to be measured over the interval from $t = 0$ to $t = 35$ in order to find upper and lower estimates within 100 grams of the actual amount of gas in the building?

low $(171-125)\Delta t \le 100$

$\longrightarrow \quad \Delta t \le \dfrac{100}{46} \approx 2.17.$

Measurements should be made every ~ 2 hours & 10 minutes in order to have estimates within 100 grams.

(7.) The function $f(x) = x\,ln(x)$ has one critical point on the interval (0, 5).

(a) (4 pts) Determine the **exact** x value (i.e., not a decimal approximation) for the location of this critical point.

$$f'(x) = \ln(x) + x\left(\tfrac{1}{x}\right) = \ln(x) + 1$$

$f'(x) = 0$ if $\ln x = -1$ so $x = e^{-1} = \frac{1}{e}$

$x =$ e^{-1} or $x = \frac{1}{e}$

(b) (3pts) Is this point a maximum or a minimum or neither of these? Explain and show your work. [Use either 1st or 2nd deriv test -- or a good graphical argument.]

By 2nd deriv. test: $f''(x) = \frac{1}{x}$; $f''\left(\frac{1}{e}\right) > 0$ so $x = \frac{1}{e}$ is a local minimum

(c) (2 pts) Determine the instantaneous rate of change of f at $x = 1$ and at $x = 2$.

$f'(1) = \ln(1) + 1 = 1$

$f'(2) = \ln(2) + 1 \approx 1.693$

@ $x = 1$, $f'(1) = 1$

@ $x = 2$, $f'(2) \approx 1.693$

(d) (2 pts) What do the values in part (c) suggest about the concavity of the function between $x = 1$ and $x = 2$? Explain.

Since f' increases between $x=1$ & $x=2$, this suggests the function is concave up between $x=1$ & $x=2$.

(e) (3 pts) Determine an equation of the tangent to the graph of f at $x = 1$.

@ $x = 1$, $f(1) = 1 \cdot \ln(1) = 0$; $m = f'(1) = 1$,

so $y - 0 = 1(x-1)$

$y = x - 1$

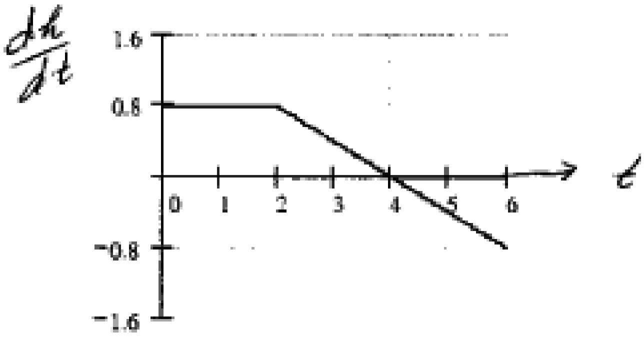
(f) (2 pts) Use your equation from part (e) to approximate $f(2)$.

Using part (e),
$f(2) \approx 2 - 1 = 1$.

[Note: $f(2) = 2\ln(2) \approx 1.386$]

(g) (2 pts) Should your estimate be an underestimate or an overestimate? Underestimate Why? — Because if the function is concave up, the tangent lies below the function.

(8.) The graph in the figure below is the graph of $\frac{dh}{dt}$, where h is the altitude in thousands of feet above sea level and t is in hours, for Professor Bob's recent climb to the top of Bear Peak in Colorado. Use the graph to answer the following questions.



(a) (3 pts) How long did it take Bob to reach the peak of the mountain? 4 hours

(b) (5 pts) What was the total change in altitude between $t = 0$ and $t = 4$?

Represented by the area under the curve $= A_{rec.} + A_\Delta = 2(0.8) + \frac{1}{2}(2)(0.8)$

$= 2.4$ thousand feet - or 2400 feet

(c) (4 pts) If Bob began his climb at 6000 feet above sea level, how high is the peak above sea level?

Height of peak $= 6000 + 2400 = 8400$ feet

[Note: $f(4) = f(0) + \int_0^4 f'(t)\,dt = 6000 + 2400$]

(d) (4pts) After 6 hours, Bob stopped at a lookout point to have a snack. What was the altitude of the lookout point?

Bob descended 800 feet from the peak, so his altitude was $8400 - 800 = 7600$ feet

(9.) (3 pts) Use the Fundamental Theorem of Calculus to evaluate the function below. To get credit, you must **show all of your work.** Please circle your answer.

[Note: This is a different problem from above.]

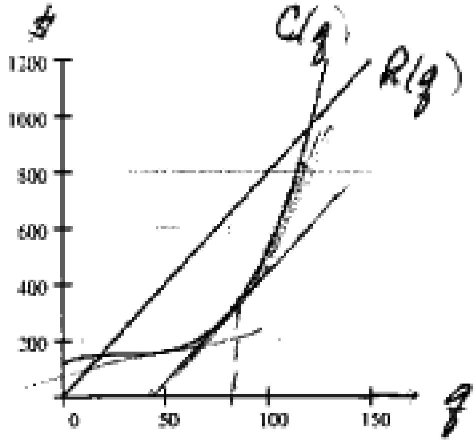
$\int_2^5 (3x^2 - 4x + 1)dx =$

$$\frac{3x^3}{3} - \frac{4x^2}{2} + x\Big]_2^5 = x^3 - 2x^2 + x\Big]_2^5$$

$= (125 - 50 + 5) - (8 - 8 + 2)$

$= 80 - 2 = 78$

(10.) Cost and revenue functions for a charter bus company are shown in the figure below, where q is the number of buses that the company owns.



(a) **(4 pts)** Should the company add a 50[th] bus? How about a 100[th]? Explain your answers using marginal revenue and marginal cost. (You may illustrate your reasons graphically as well, if you like.)

The company should add the 50th bus, because at $q = 50$ MC (or the slope of $C(q)$) is less than MR -- ie, revenue is increasing faster than cost. The company should not add the 100th bus because @ $q = 100$, costs are increasing at a faster rate than revenue -- ie $MR < MC$.

(b) **(3 pts)** What does $C'(50) = A$ (A, a constant) mean in the context of this problem? What are the units of the 50 and the units of A?

In the expression $C'(50) = A$, the units of 50 are buses & the units of A are dollars per bus. The expression represents the rate at which costs are increasing once the company has 50 buses. In economic terms, approximately the cost of going from 50 - 51 buses is $A.

(c) **(4 pts)** Estimate the number of buses the company should have in order to maximize profit. Explain how you determined your estimate.

Profit will be maximized when $R(q) - C(q)$ is greatest -- which also (by calculus) occurs when $R'(q) = C'(q)$. This means the slope of the tangent to $C(q)$ is equal to the slope of $R(q)$. This appears to be around $q = 80$. [See graph.]

(Problem 10 continued)

(d) (6 pts)

(i) If the average cost, $a(q)$, is given by $a(q) = \dfrac{C(q)}{q}$, approximate q_0 so that $a(q_0)$ is the minimal average cost.

From the graph, minimum average cost appears to be when $q \approx 60$.

(ii) Show ***analytically*** that average cost will be minimized when $C'(q) = a(q)$.

Given $a(q) = \dfrac{C(q)}{q}$, then

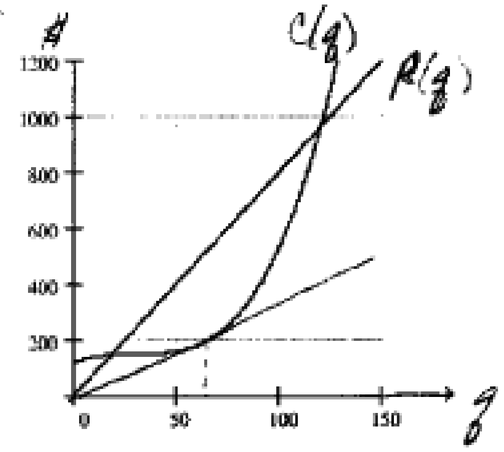
$$a'(q) = \frac{qC'(q) - C(q)(1)}{q^2} \quad \& \quad a'(q) = 0 \text{ if}$$

$$qC'(q) = C(q)$$

$$\rightarrow \quad C'(q) = \frac{C(q)}{q} = a(q)$$

Note: $a(q) = \dfrac{C(q)}{q}$ can be visualized as slope from $(0,0)$ to $(q, C(q))$. These slopes decrease for $0 < q < q_0$ & then increase for $q > q_0$. Thus, there is a min @ $q = q_0$.

(iii) Demonstrate on the graph below how this result can be shown graphically.



(11.) And, for good measure, one last derivative…. No need to simplify, but **show all your work.**

(3 pts) Find the derivative of $k(t) = \dfrac{(3t-4)}{\cos(2t)}$.

ok to here

$$k'(t) = \frac{\cos(2t)(3) - (3t-4)(-\sin 2t)(2)}{\cos^2(2t)}$$

$$\underline{\text{or}} \quad = \frac{3\cos(2t) + (6t-8)\sin(2t)}{\cos^2(2t)}$$

If use Quot Rule get

$k'(t) = (3t-4)\left[-(\cos(2t))^{-2}(-2\sin(2t))\right] + 3(\cos(2t))^{-1}$

[One more page…]

(12.) The city council is planning to construct a new sports ground in the shape of a rectangle with semicircular ends. A running track 400 meters long is to go around the perimeter.

(a) **(6 pts)** What choice of dimensions will make the rectangular area in the center as large as possible?

Given:

$P = 2l + \pi w = 400$

so $l = \frac{400 - \pi w}{2}$

$= 200 - \frac{\pi}{2} w$

We want to Maximize $A = l \cdot w$

In one variable: $A = \left(200 - \frac{\pi}{2} w\right) \cdot w$

$= 200w - \frac{\pi}{2} w^2$

So, $A' = 200 - \pi w$ & $A' = 0$ if $w = \frac{200}{\pi}$ meters

Note: $A''\left(\frac{200}{\pi}\right) = \frac{200}{\pi}(-\pi) = -200 < 0$, so the only crit pt, $w = \frac{200}{\pi}$ is a max.

Then $l = 200 - \frac{\pi}{2}\left(\frac{200}{\pi}\right)$

$l = 100$ meters

(b) **(4 pts)** What should the dimensions be if the total area enclosed by the running track is to be as large as possible?

If total area is to be maximized then we want to maximize

$TA = l \cdot w + \pi\left(\frac{w}{2}\right)^2 = lw + \frac{\pi w^2}{4}$

Then $TA = \left(200w - \frac{\pi}{2} w^2\right) + \frac{\pi w^2}{4} = 200w - \frac{\pi w^2}{4}$

& $TA' = 200 - \frac{\pi w}{2}$. Then $TA' = 0$ if

$\frac{400}{\pi} = w$ meters

Note: $TA'' = -\frac{\pi}{2} < 0$ for all w, so $w = \frac{400}{\pi}$ is a max. & is the global max. When $w = \frac{400}{\pi}$, $l = 0$ meters, so the track is a circle.

Good luck, and a happy spring and summer to each of you!