Solution Key

MATH 115 — MIDTERM EXAM # 2

DEPARTMENT OF MATHEMATICS
University of Michigan

March 20, 2002

NAME: ______________________ ID NUMBER: ______________________

SIGNATURE: ______________________________

INSTRUCTOR: ________________ SECTION NO: ______________________

General Instructions: Do not open this exam until you are told to begin. This test consists of 12 questions on 10 pages (including this cover sheet). The last page is blank and is for your use as a worksheet. The exam is worth 100 points. Do not separate the pages of exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

You are allowed two sides of a 3 by 5 card of notes and are expected to use your calculator.

PROBLEM	POINTS	SCORE
1	12	
2	7	
3	7	
4	8	
5	9	
6	10	
7	10	
8	8	
9	4	
10	7	
11	9	
12	9	
TOTAL	100	

1. (2 points each) **True or False.** Circle True only if the statement is always true.

(a) If f' is increasing, then f is increasing. T F

(b) If f is an exponential function, then $\frac{d}{dx}\ln f(x)$ is constant. T F

(c) If $f''(x) = 0$ for all x, then f is a constant function. T F

(d) There is a function f so that $f(x) > 0$, $f'(x) < 0$, and $f''(x) < 0$ for all x. T F

(e) If $f''(x) < 0$ for all x, then $f(x) \le f(0) + f'(0)x$ T F

(f) If $f'(x) = 0$, then f has either a relative maximum or relative minimum at x. T F

2. (7 points) The function g has a continuous derivative whose values are given in the following table. There is no more than one critical point of g between any two consecutive x-values in the table.

Note that the table gives values of $g'(x)$, **NOT** $g(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
$g'(x)$	-9	-2	2	1	-1	-3	-6	-5	-4	2	10

(a) Estimate the x-coordinates of the critical points of g for $0 < x < 10$.

$1 < x < 2$ or $x \approx 1.5$
$3 < x < 4$ or $x \approx 3.5$
$8 < x < 9$ or $x \approx 8.5$

(b) For each critical point found in part (a), determine if it corresponds to a local maximum or minimum of the function g. Be sure to explain.

For $x \approx 1.5$, there is a local min because g decreases to the left of the CP and increases to the right

For $x \approx 3.5$, there is a local max because g increases to the left of the CP + increases to the right.

For $x \approx 8.5$, there is a local min since g decreases to the left of the CP + increases to the right.

OR: If the temp increases to 24°, my bill will decrease approx 17 cents.

3. (7 points) Suppose that $f(T)$ is the cost to heat my house, in dollars per day, when the outside temperature is T degrees Fahrenheit.

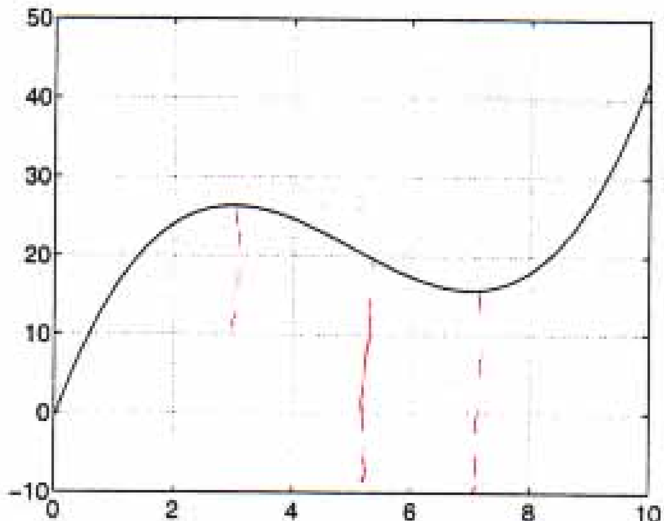
(a) What does $f'(23) = -0.17$ mean in the context of this problem?

When the temperature is 23°F, my costs are decreasing at the rate of approximately 17¢/day per degree.

(b) If $f(23) = 7.54$, and $f'(23) = -0.17$, what is the approximate cost to heat my house when the outside temperature is 20 degrees Fahrenheit?

$$f(20) \approx f(23) - 0.17(20-23)$$
$$= 7.54 + 0.17(3) = \$8.05 \text{ per day}$$

4. (8 points) An object is moving on a straight line so that its distance (measured in feet) to the right of a fixed point on the line at time t (measured in seconds) is given by the function s whose graph is in the following figure.



Graph of s(t)

(a) At what times (approximately) is the object moving to the right? to the left?

The object is moving to the right for $0<t<3$ and $7<t<10$. To the left for $3<t<7$.

(b) At what times (approximately) does the object have positive acceleration? negative acceleration? (Explain what properties of the graph give you this information.)

Acceleration is positive when the second derivative is positive. This is when the graph is concave up or for $\sim 5<t<10$. Accel. is neg when graph conc. down on $0<t<5$.

(c) At what times (approximately) is the velocity of the object increasing? Explain.

Velocity is ~~[illegible]~~ increasing when acceleration is positive ... ie, $5<t<10$.

5. (9 points) Find the equation of the tangent line to the curve $2x^2y^2 - x^3 - y^5 + 1 = 0$ at the point $(2, 1)$.

✓ing point: $8-8-1+1 \stackrel{\checkmark}{=} 0$

Differentiating:

$$2x^2\left(2y\frac{dy}{dx}\right) + 4xy^2 - 3x^2 - 5y^4\frac{dy}{dx} = 0$$

using $(2,1)$

$$8\left(2\frac{dy}{dx}\right) + 8 - 12 - 5\frac{dy}{dx} = 0$$

$$11\frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{4}{11}$$

$$y - 1 = \frac{4}{11}(x-2)$$

$$y = \frac{4}{11}x - \frac{8}{11} + 1$$

$$y = \frac{4}{11}x + \frac{3}{11}$$

or equiv.

6. (10 points) **(a)** Find the Taylor polynomial of degree two that approximates the function $(1+2x)^{\frac{3}{2}}$ at $x = 0$ (Show your work!).

$f(0) = (1)^{3/2} = 1$

$f'(x) = \frac{3}{2}(1+2x)^{1/2}(2) = 3(1+2x)^{1/2}$

$f'(0) = 3$

$f''(x) = \frac{3}{2}(1+2x)^{-1/2}(2)$

$f''(0) = 3$

$P_2(x) = 1 + 3x + \frac{3}{2}x^2$

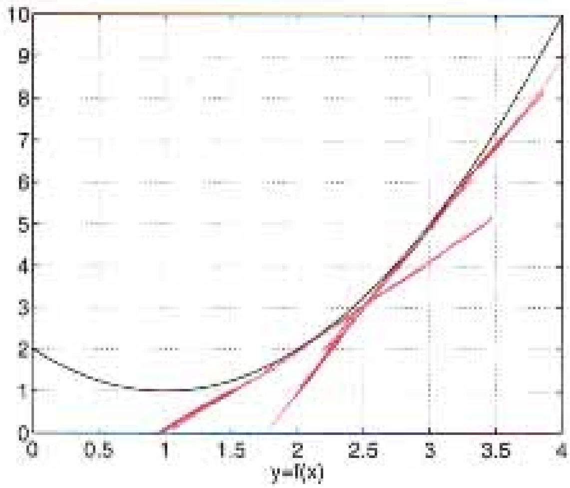
(b) What is the local linearization of $(1+2x)^{\frac{3}{2}}$ near $x = 0$?

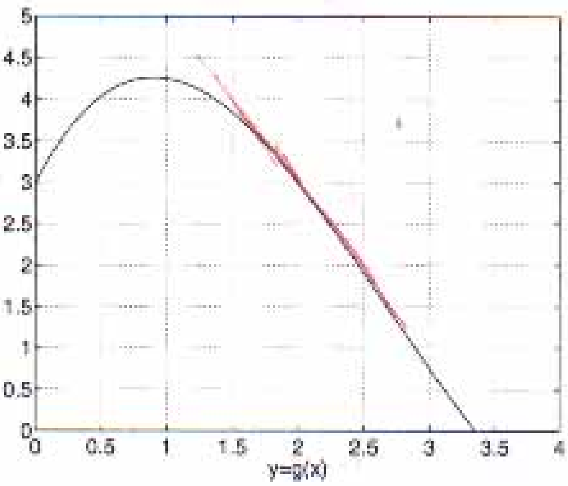
$y = 1 + 3x$

(c) Is the local linearization of $(1+2x)^{\frac{3}{2}}$ an overestimate or underestimate of the function? Why?

The local linearization is an underestimate because $f''(0) > 0$, so the function is concave up there. In fact, f is concave up for all x, so a linear approximation is an underestimate for all x... (for which the function is defined...)

7. (10 points) Let f and g be functions with the following graphs.





Use the graphs to estimate each of the following derivatives. Show your work and circle your answers.

(a) $h'(2)$ if $h(x) = f(x)g(x)$

$$h'(x) = f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

$$h'(2) = 2(-2) + 2(3) = 2$$

$f(2) = 2$

$f'(2) \approx 2$

$g(2) = 3$

$g'(2) \approx -\frac{1}{1/2} = -2$

(b) $h'(2)$ if $h(x) = \frac{f(x)}{g(x)}$

$$h'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

$$h'(2) = \frac{3(2) - 2(-2)}{(3)^2} = \frac{10}{9}$$

(c) $h'(2)$ if $h(x) = f(g(x))$

$$h'(x) = f'(g(x))\cdot g'(x)$$

$$h'(2) = f'(3)\cdot g'(2)$$

$$= 4(-2) = -8$$

$g(2) = 3$

$f'(3) \approx \frac{2}{\frac{1}{2}} = 4$

8. (8 points) On the set of axes provided, draw the graph of a smooth function f such that this function has all of the following properties.

(a) $f(3) = 2$

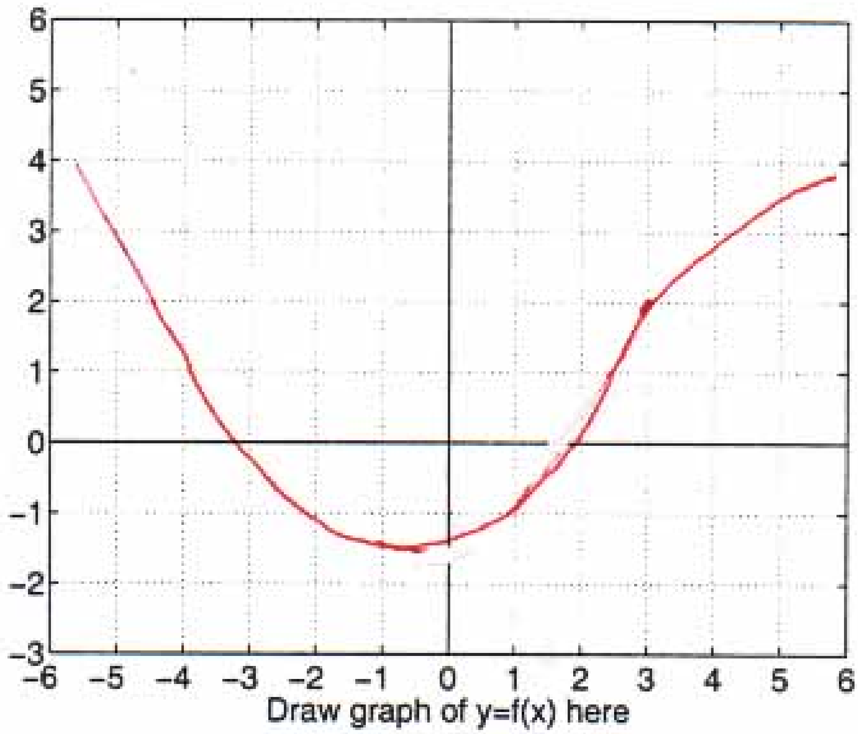
(b) $f' < 0$ for $x < 0$

(c) $f' > 0$ for $x > 0$

(d) $f'' > 0$ for $x < 3$

(e) $f'' < 0$ for $x > 3$

(f) the graph of f does not pass through the origin



(b) Is it possible that $f(x) = 0$ for some $x > 3$? Explain.

No. Since $f'(x) > 0$ for $x > 0$, the function increases for $x > 0$. Thus, $f(x) > 2$ for all $x > 3$.

9. (4 points) Let f, g be functions such that $f''(x) > 0$ and $g''(x) < 0$ for all x. In how many points can the graphs of f and g intersect? Circle all possible answers.

(i) no points

(ii) 1 point

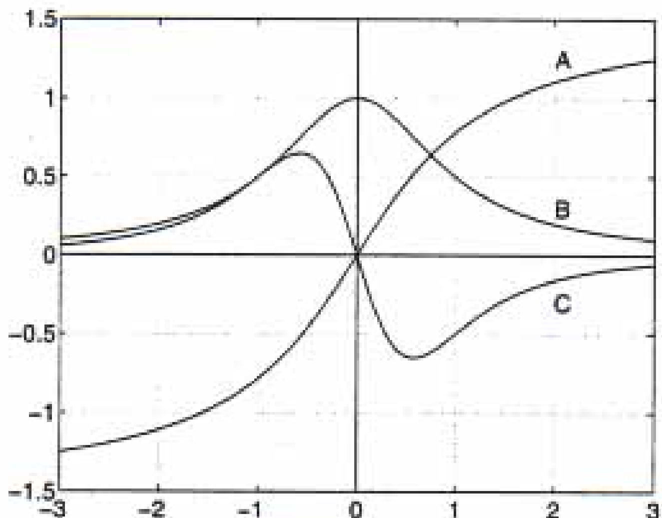
(iii) 2 points

(iv) 3 points

(v) infinitely many points

NPC

10. (7 points) (a) The figure below shows graphs of a function f and its first and second derivatives, f' and f''. Identify by the label on the graph which function is f, which is f', and which is f''.



Graphs of f, f', and f''

A is the graph of f

B is the graph of f'

C is the graph of f''

(b) Give a clear explanation of your reasoning for the choices you made in part (a).

Graph A cannot be the derivative of any other function since all functions are increasing for $x<0$. Thus, A is f. Since A is increasing for all x, & B is positive for all x, B is f'. Graph C is positive where B is increasing (& where A is concave up) & C is negative when B is decreasing (& where A is concave down, so C is f''.

11. (9 points) Recall Hanky- and Pankytowns? On the first exam, we saw that the population of Pankytown, in thousands, could be modeled by

$$P(t) = 50(0.8)^t$$

where t is the number of months after February, 2001 when valentines were banned in Pankytown.

(a) At what rate was the population of Pankytown changing in May of 2001?

$$P'(t) = 50(\ln(0.8))\cdot(0.8)^t$$

$$P'(3) = 50(\ln(0.8))(0.8)^3$$

$$\approx -5.712$$

The population is decreasing at approx 5712 people per month.

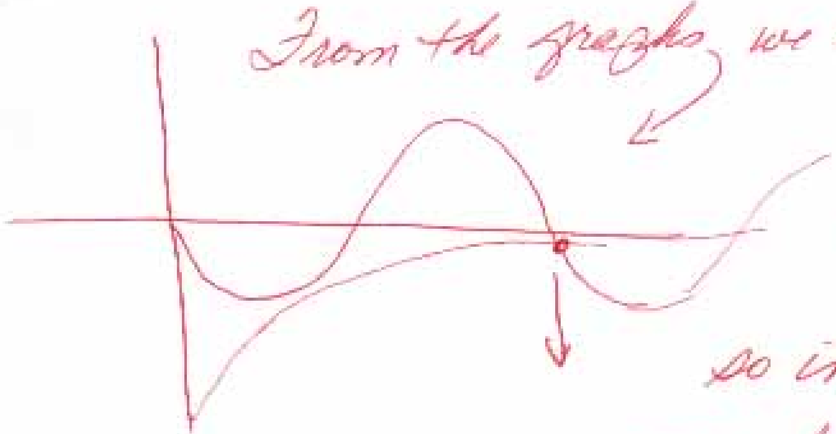
(b) We also found in Exam 1 that the population of Hankytown (in thousands) was given by

$$H(t) = 9\cos\left(\frac{\pi t}{6}\right) + 11$$

with $t = 0$ representing the month of February, 2001. Is there a time (or times) during the first 18 months after February, 2001, that the models indicate that the populations of Pankytown and Hankytown are changing at the same rate? If so, when? If not, explain why not. Clearly explain how you found your answer.

$$H'(t) = \frac{-3\pi}{2}\sin\left(\frac{\pi t}{6}\right)$$

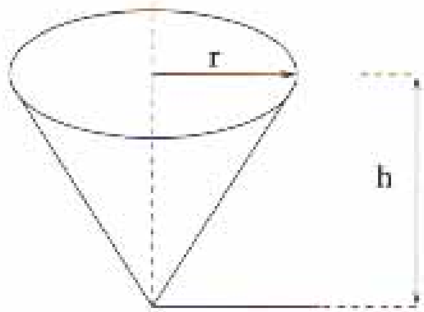
From the graphs, we see



The rates are the same @ $t \approx 12.3$, so in Feb. 2002 the populations are changing at the same rate

Also, for $t \approx 19.9$ so again in July, 2002

12. (9 points) Fluid flows out of the bottom of a cone-shaped vessel at the rate of 3 cubic cm per second (see figure below). If the radius of the cone is one-third of its height, how fast is the height of the fluid changing when the fluid is 6 cm deep in the center of the cone. Be sure to show your work and give the correct units in your answer. (Remember that the volume of a cone is $\frac{1}{3}\pi r^2 h$.)



Given:

$$V = \frac{1}{3}\pi r^2 h$$

and $r = \frac{1}{3}h$

$$\frac{dV}{dt} = -3\,\frac{cm^3}{sec}$$

we can write

$$V(h) = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 \cdot h = \frac{\pi}{3}\left(\frac{h^3}{9}\right)$$

So

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(\frac{1}{3}h^2\right)\frac{dh}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

WWTF

$\frac{dh}{dt}$ when $h = 6$ cm, using $\frac{dV}{dt} = -3$

$$-3 = \frac{1}{9}\pi(36)\cdot\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-3}{4\pi} \text{ cm/sec}$$