

MATH 115 — FINAL EXAM WITH SOLUTIONS

DEPARTMENT OF MATHEMATICS
University of Michigan

April 23, 2002

NAME: _____

ID NUMBER: _____

SIGNATURE: _____

INSTRUCTOR: _____

SECTION NO: _____

General Instructions: Do not open this exam until you are told to begin. This test consists of 11 questions on 12 pages (including this cover sheet). The last page is blank and is for your use as a worksheet. The exam is worth 100 points. Do not separate the pages of exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

You are allowed two sides of a 3 by 5 card of notes and are expected to use your calculator.

PROBLEM	POINTS	SCORE
1	14	
2	7	
3	9	
4	9	
5	10	
6	10	
7	12	
8	8	
9	4	
10	5	
11	12	
TOTAL	100	

Note: In this version of the test, there are references to problems from the text. These are similar to the test problem and are good for further self study. **ALL** of these problems were assigned homework problems in the term the test was given. The text for the course in this term was *CALCULUS*, by Hughes-Hallet, Gleason, McCallum, et. al., Third Edition.

1. (14 points) **(a)** (See the odd exercises in Chapter 6, Section 2.) An antiderivative of

$$f(t) = \sin(t) + \frac{1}{\cos^2 t}$$

is $F(t) = \underline{-\cos t + \tan t}$.

- (b)** If $\int_0^3 f(x) dx = 3$, then $\int_0^3 (f(x) + 2) dx = \int_0^3 f(x) dx + \int_0^3 2 dx = 3 + 2 \cdot 3 = 9$.

- (c)** A cubic polynomial (3rd degree polynomial) always has a

(i) local maximum	Yes	<u>No</u>
(ii) local minimum	Yes	<u>No</u>
(iii) global maximum	Yes	<u>No</u>
(iv) global minimum	Yes	<u>No</u>
(v) inflection point	<u>Yes</u>	No

- (d)** (See problem 79, Chapter 6, Section 2.) The exact value of c such that

$$\int_0^c x\sqrt{x} dx = \frac{4}{5}.$$

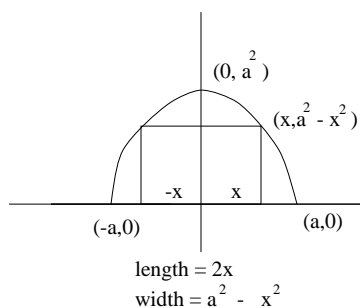
is $c = 2^{\frac{2}{5}} = 4^{\frac{1}{5}}$

Calculation:

$$\frac{4}{5} = \int_0^c x\sqrt{x} dx = \int_0^c x^{\frac{3}{2}} dx = \frac{2}{5} x^{\frac{5}{2}} \Big|_0^c = \frac{2}{5} \left(c^{\frac{5}{2}} - 0 \right) = \frac{2}{5} c^{\frac{5}{2}} \quad \text{or} \quad \frac{4}{5} = \frac{2}{5} c^{\frac{5}{2}}.$$

Therefore, $c^{\frac{5}{2}} = 2$ **or** $c = 2^{\frac{2}{5}}$.

2. (7 points) What is the largest area a rectangle can have if its base lies on the x -axis and its upper vertices lie on the curve $y = a^2 - x^2$? (Your answer will be in terms of a . Show your work.).



Solution: The rectangle has length $l = 2x$ and width $w = a^2 - x^2$, so we want to maximize the area

$$A = lw = 2x(a^2 - x^2) = 2a^2x - 2x^3, \quad \text{for } 0 \leq x \leq a.$$

Because $A = 2a^2x - 2x^3$, we have $A' = 2a^2 - 6x^2$, so that if $A' = 0$ then $x^2 = \frac{a^2}{3}$ and $x = \frac{|a|}{\sqrt{3}} = \frac{a}{\sqrt{3}}$ since $x \geq 0$.

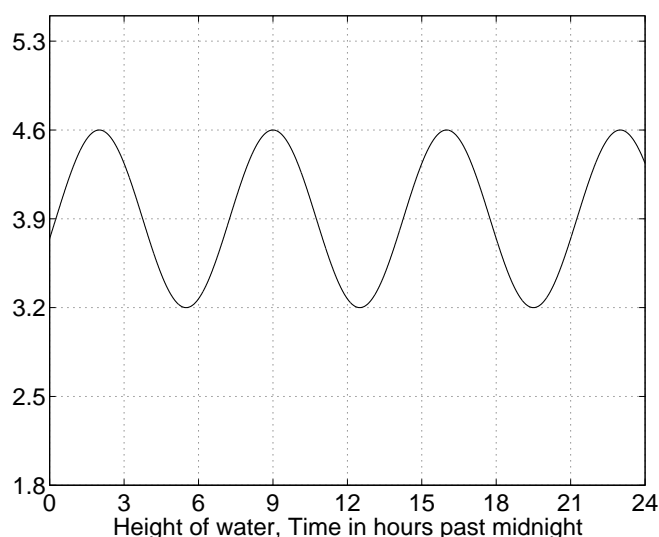
Note that this is the only critical point of A for $0 \leq x \leq a$. At the endpoints of this interval, $A(0) = 0$ and $A(a) = 0$. The value of A at the critical point is positive, $A\left(\frac{|a|}{\sqrt{3}}\right) = \left(2\frac{a \cdot a^2}{\sqrt{3}} - 2\left(\frac{a}{\sqrt{3}}\right)^3\right) = \frac{2a}{\sqrt{3}}\left(a^2 - \frac{a^2}{3}\right) = \frac{2a}{\sqrt{3}}\left(\frac{2a^2}{3}\right) = \frac{4a^3}{3\sqrt{3}} > 0$. Since there is only one critical point in the interval, this value of A must be the global maximum for $0 < x < a$.

3. (9 points) (See Team homework problem 34, Chapter 1, Section 5.) The water level in an an underground tank varies periodically every 7 hours, oscillating between a maximum level of 4.6 feet and a minimum of 3.2 feet.

(a) If the water reaches a maximum height at 9am on a certain day, write a formula using the sine or cosine function, for the height h as a function of time t , where t is measured in hours past midnight of that day.

The amplitude $|A|$ must be equal to $(4.6 - 3.2)/2 = 1.4/2 = .7$ (feet) and the midline must be $3.2 + .7 = 4.6 - .7 = 3.9$ (feet), and the period is 7 (hours), so the height in feet is given by

$$h(t) = .7 \cos \left(\frac{2\pi}{7}(t - 9) \right) + 3.9$$



(b) What are the period and amplitude of your function? (Use units, if appropriate.)

Period = .7 hours

Amplitude = .7 feet

(c) At what rate is the water rising or falling (indicate which) at 2pm on that day? (Be sure to use units in your answer.)

$$h'(t) = -.7 \sin \left(\frac{2\pi}{7}(t - 9) \right) \left(\frac{2\pi}{7} \right)$$

At 2pm, $t = 14$, so

$$h'(14) = -.2\pi \sin \left(\frac{2\pi}{7}(5) \right) \simeq .6126 \text{ ft./hour}$$

At 2pm, the water is rising at the rate of approximately .6126 feet per hour.

4. (9 points) (See problem 3, Chapter 5, Section 1.) Last week David ran the Naked Mile, starting out at a fast pace with the idea of winning the race. His friend, John rode along on his bike to clock David's times. However, at the end of the race the police were chasing David so that he kept on running to avoid being arrested. John followed and recorded David's speeds at 5 minute intervals. David was slowing down all the time, but fortunately for him, the policemen were unable to catch him. They finally gave up chasing him after 25 minutes. David continued for an additional five minutes before stopping.

The speeds John clocked are recorded in the following table. In recounting his experience, David wondered how far he actually ran in the half hour. Help him out by answering the questions in parts (a) and (b). (Be sure to show your work when answering the questions).

time (in minutes):	0	5	10	15	20	25	30
speed (in miles per minute):	.2	.16	.14	.12	.12	.1	.05

(a) Assuming that David's speed never increases throughout the run, use the data in the table to determine the best estimate for the total distance that David ran during the 30 minutes.

Solution: Since David's speed is never increasing, the left hand sum gives an upper estimate for the distance he travelled,

$$\text{Upper est} = (.2 + .16 + .14 + .12 + .12 + .1)(5) = 4.2 \text{ miles}$$

and the right hand sum gives a lower bound,

$$\text{Lower est} = (.16 + .14 + .12 + .12 + .1 + .05)(5) = 3.45 \text{ miles}$$

The best estimate, given the above information, is the average of the upper and lower bounds,

$$\frac{4.2 + 3.45}{2} = 3.825 \text{ miles}$$

(b) To be sure of estimating the distance David traveled to within .15 miles, how frequently would John have needed to record the measurements of David's speed?

Solution: We want the error to be at most .15 miles. Since the difference between the estimates given by the upper and lower sums with measurements taken Δt minutes apart is $(2 - .05)\Delta t$, we want $.15\Delta t \leq .15$ or $\Delta t \leq 1$ minute. Therefore, if John had taken the measurements every one minute, then we could have been sure of this estimate, no matter which of the estimates obtained in part (a) was used.

If one uses the average of the left and right hand sums for the estimate, however, you can do a little better. In this case, the error taken with measurements Δt minutes apart is $\frac{(2 - .05)\Delta t}{2}$, so to make the error at most .15 miles, it suffices to take $.15\Delta t/2 \leq .15$ or $\Delta t \leq 2$. Therefore, John would only have to measure David's speed every two minutes to achieve this accuracy.

5. (10 points) (See problem 11, Chapter 4, Section 2.) Consider the family of functions defined for $x \geq 0$ by $y = axe^{-bx}$ where a and b are positive numbers. Find all members of this family that pass through the point $(2, 3)$ and have a critical point at $x = 2$. Determine if this critical point is a local maximum, local minimum, or neither. If it is a local maximum or minimum, is it also a global maximum or minimum on this domain (i.e. for $x \geq 0$)?

Solution: Let $y = f(x) = axe^{-bx}$. We are given that $f(2) = 3$ and $f'(2) = 0$. First calculate $f'(x)$.

$$f'(x) = ax(-be^{-bx}) + ae^{-bx} = ae^{-bx}(-bx + 1)$$

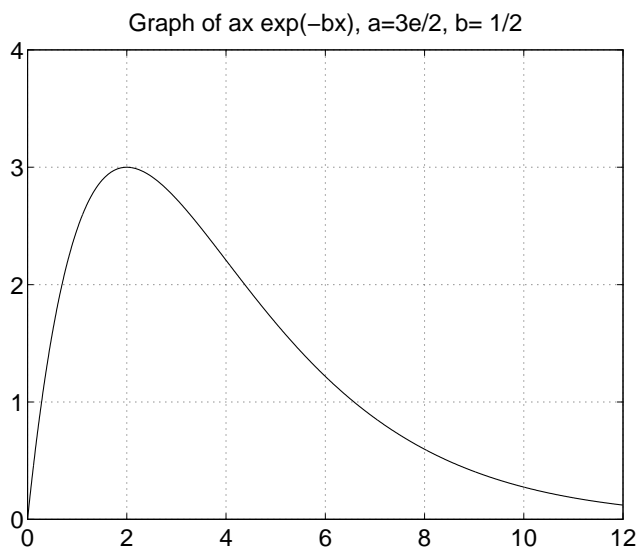
If $f'(x) = 0$, then we must have $bx = 1$. Since this holds for $x = 2$, $2b = 1$ or $b = \frac{1}{2}$.

Next use $f(2) = 3$ and $b = \frac{1}{2}$ to see that

$$3 = f(2) = 2ae^{-1} = \frac{2a}{e} \quad \text{or} \quad a = \frac{3e}{2}.$$

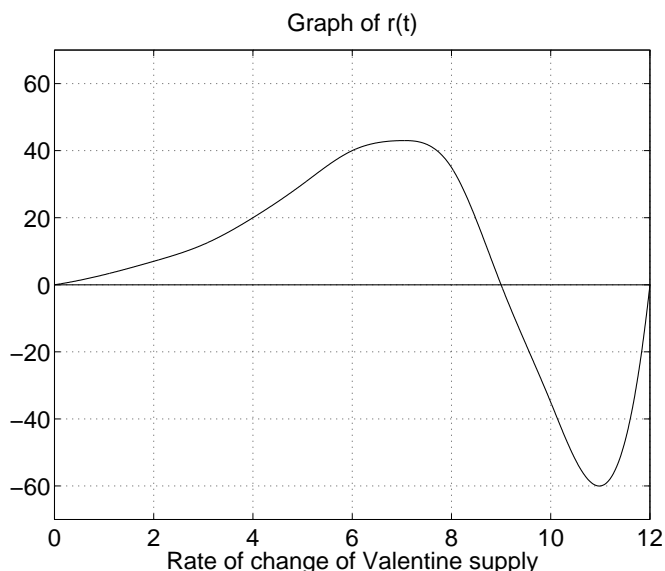
Thus, $y = \frac{3e}{2}xe^{-\frac{1}{2}x} = \frac{3}{2}xe^{(1-\frac{x}{2})}$.

Note: We saw that $x = 2$ is the only critical point of the function. Also, $f'(x) > 0$ for $x < \frac{1}{b}$, (or $x < 2$) and $f'(x) < 0$ for $x > \frac{1}{b}$, (or $x > 2$) from the formula derived for $f'(x)$. Thus, $x = \frac{1}{b}$ (or $x = 2$) gives a local maximum. Since $f(x) \geq 0$, $f(0) = 0$, and $f(x) \rightarrow 0$ as $x \rightarrow +\infty$, $x = \frac{1}{b}$ (or $x = 2$) is also a global maximum for $x \geq 0$.



6. (10 points) (See problem 17 and Team problem 26, Chapter 5, Section 3.) Recall that Hankytown, the community famed for making valentines, has a fluctuating population based on the influx of migrant valentine makers. The number of valentines in the city coffers varies also according to the season of the year. The graph below shows the rate, $r(t)$ (in 1000's of valentines per month), at which the supply of valentines changes over a 12 month period, where

$t = 0$ corresponds to the beginning of January.



(a) Over what period of time did the valentine supply grow? Solution: The supply grows while $r(t) > 0$ so for about $0 \leq t \leq 9$. This corresponds to the time from January 1 to the beginning of October.

(b) When was the supply of valentines growing most rapidly?

Solution: The supply was growing most rapidly where $r(t)$ takes its maximum value, or at about $t = 7$, the beginning of August.

(c) Write a mathematical expression giving the average rate at which the valentine supply is changing over the first four months shown in the graph.

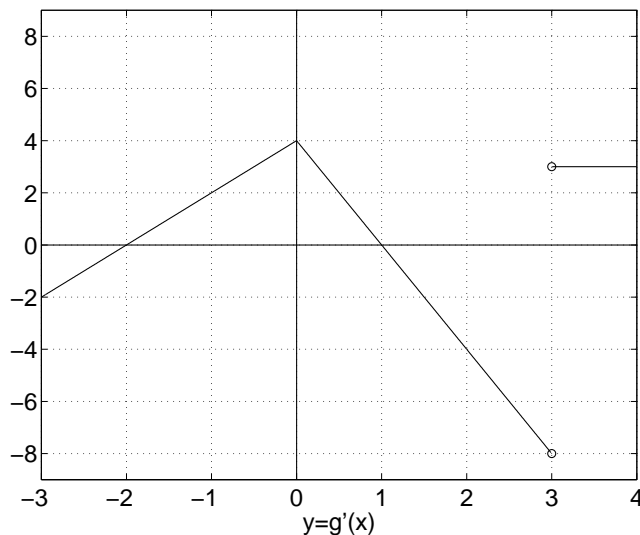
Solution: The average value of $r(t)$ over the first four months of the year, $0 \leq t \leq 4$, is

$$\frac{1}{4} \int_0^4 r(t) dt$$

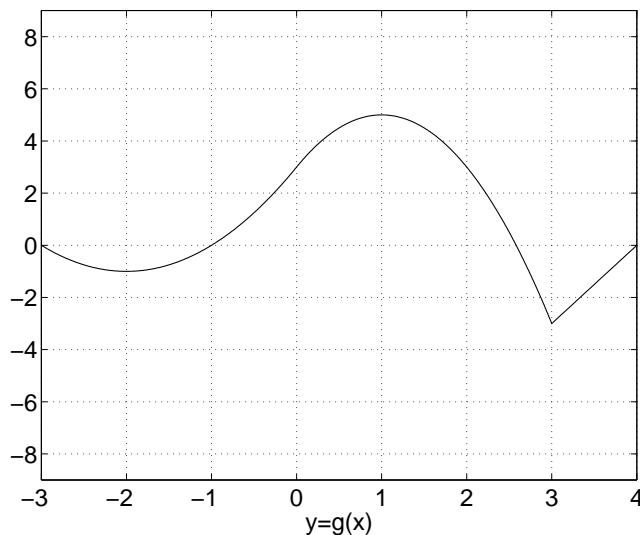
(d) Given that there were 25,000 valentines in the warehouse at the beginning of the period shown, write a mathematical expression for the total number of valentines in the warehouse at the end of the 12 month period. Were there more or fewer than 25,000 valentines at this time. How do you know?

Solution: The number of valentines in the warehouse at the end of the 12 month period (in thousands of valentines) $= 25 + \int_0^{12} r(t) dt$. From the graph, it appears that the area under the curve $r(t)$ and above the t -axis is greater than the area above the curve $r(t)$ but below the t -axis. Therefore, $\int_0^{12} r(t) dt > 0$, so there are a greater number of valentines in the warehouse at the end of the period than at the beginning.

7. (12 points) (See Team problem 18 and problem 15 from Chapter 6, Section 1.) A function g is known to be continuous and the graph of its derivative, g' , for $-3 \leq x \leq 4$ is given in the following figure.



(a) Given that $g(-3) = 0$, sketch the graph of g on the axes provided below. In the space below the figure, give the coordinates of **ALL** *critical points* of g .



(b) Critical Points:

$$(-2, -1), \quad (1, 5) \quad (3, -3)$$

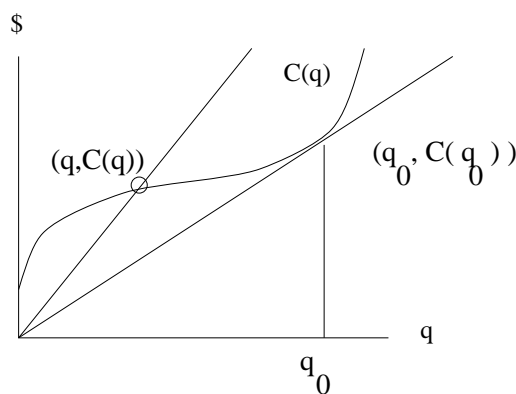
8. (8 points) (See Team problem 16, Chapter 4, Section 4, as well as problems 29 and 31 from Chapter 4, Section 3.) (a) It is a fact from economics that when the average cost $C(q)/q$ of producing $q > 0$ units of a quantity is a minimum, then this average cost is equal to the marginal cost. Show analytically why this is so.

Solution: At a point where the average cost $a(q) = C(q)/q$ is a minimum the derivative of $a(q)$ must be equal to 0. By the quotient rule,

$$a'(q) = \frac{qC'(q) - C(q)}{q^2}, \quad q > 0$$

so $a'(q) = 0$ if and only if $qC'(q) - C(q) = 0$, or $C'(q) = \frac{C(q)}{q}$. Thus, at the critical points of $a(q)$, the marginal cost, $C'(q)$, is equal to the average cost $\frac{C(q)}{q}$.

(b) Using the graph of $C(q)$ shown below, a typical cost function as drawn in the text, indicate on the q -axis the value of q_0 which minimizes the average cost. Explain graphically why the average cost is equal to the marginal cost at this point.



Line through $(q, C(q))$ has slope equal to average cost.

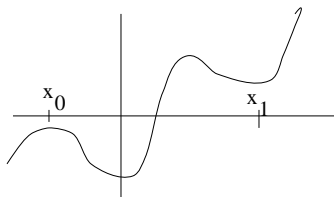
Line with minimum slope is tangent to graph of $C(q)$.

The average cost $a(q) = \frac{C(q)}{q}$ is the slope of the line from the origin to a point $(q, C(q))$ on the graph of C . The line with minimal slope is shown above. At this point, q_0 , the line from the origin is tangent to the curve so its slope is also equal to $C'(q_0)$, the marginal cost at that point.

9. (4 points) Explain either why the following statement is always true or show a function for which it is false.

“If f is a differentiable function defined for all x and if f has a local maximum at x_0 and a local minimum at x_1 , then $f(x_0) \geq f(x_1)$. ”

False. One example is given by the following graph.



10. (5 pts) Upon returning home this summer, you meet a good friend who is just now graduating from high school. He has done very well in precalculus and has a good understanding of the graphs, tables, and formulas that a good student in precalculus should know. He is planning to come to U-M next year and will take Math 115. He has heard that one of the basic concepts in 115 is something called a derivative.

In the space below, explain what you would say to give your friend a good idea of what the derivative means, illustrating this in as many ways as you believe will help your friend understand the concept.

Solution: There are many good answers to this question. They may include ideas of

- slope of tangent or slope of a curve at a point
- instantaneous rate of change
- discussion of velocity vs. position at time t .
- table of values with appropriate discussion
- the definition as the limit of a difference quotient.

Note: strictly a formula (e.g. $f(x) = x^3, f'(x) = 3x^2$) or only the limit definition without further explanation is not sufficient to receive full credit.

11. (12 points) (See problems 10 and 11 from Chapter 4, Section 4.) Ms. Manufacturer, a producer of diamond-studded widgets, finds that her company can sell q widgets per week if they are priced at p each, where

$$q = 100 - 2p$$

and p is in hundreds of dollars. Her cost, also measured in hundreds of dollars, for producing q widgets is

$$C(q) = 100 + 10q + \frac{1}{2}q^2.$$

(a) How many widgets should her company manufacture each week to achieve the least cost per widget? That is, the least average cost. (Be sure to show your work.)

Solution: The average cost is $a(q) = C(q)/q = 100/q + 10 + .5q$. To minimize $a(q)$, we look for points where $a'(q) = 0$,

$$0 = a'(q) = \frac{-100}{q^2} + \frac{1}{2} \quad \text{or} \quad q^2 = 200.$$

Therefore, $q = \sqrt{200} \simeq 14.14$ is the only critical point of the average cost function for $q > 0$. Further, since the average cost $a(q)$ is positive and $a(q) \rightarrow +\infty$ as $q \rightarrow 0$ or $q \rightarrow +\infty$, this critical point must be a global minimum of $a(q)$ on $q > 0$. Thus, she should produce about 14 widgets per week to achieve the least average cost.

(b) Determine the formula for the revenue $R(q)$ received each week if q widgets are sold.

$$R(q) = \text{price} \cdot \text{quantity} = \left(50 - \frac{q}{2}\right) q = 50q - \frac{q^2}{2}$$

(c) How many widgets should Ms. Manufacturer's company produce each week in order to maximize profits? At what price should the widgets be sold?

$$\text{Profit} = \pi(q) = \text{Revenue} - \text{Cost} = \left(50q - \frac{q^2}{2}\right) - \left(100 + 10q + \frac{1}{2}q^2\right)$$

At the maximum of $\pi(q)$,

$$0 = \pi'(q) = R'(q) - C'(q) = (50 - q) - (10 + q) = 40 - 2q \quad \text{or} \quad q = 20.$$

Since $\pi''(q) = -2 < 0$ at $q = 20$, this must be a local maximum, and since it is the only critical point in $q > 0$ it must be a global maximum. The profit will be maximized when 20 widgets are produced each week and sold for a price of $50 - \frac{20}{2} = 40$, or \$4,000 apiece.