Solution Key

MATH 115 MIDTERM EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

February 12, 2003

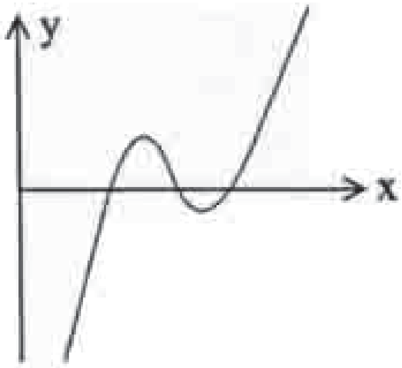
NAME: ______________________ ID NUMBER: ______________________

INSTRUCTOR: ________________ SECTION NO: ______________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 12 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it.
6. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
7. Include units in your answers where appropriate.
8. You are allowed two sides of a 3 by 5 card of notes. You may also use your calculator.
9. Please turn off all cell phones.

PROBLEM	POINTS	SCORE
1	8	
2	9	
3	3	
4	4	
5	5	
6	7	
7	10	
8	7	
9	8	
10	14	
11	11	
12	14	
TOTAL	100	

1. (8 pts.) The figures below show the graphs of four functions for positive values of x. For each of the figures, circle the function which best respresents the graph in the figure. Assume that a, b, c, $k > 0$.

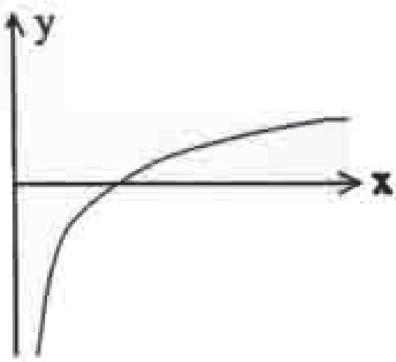


$y = -k(x+a)(x+b)(x+c)$

$y = k(x+a)(x+b)(x+c)$

$y = k(x-a)(x-b)(x-c)$

$y = -k(x-a)(x-b)(x-c)$

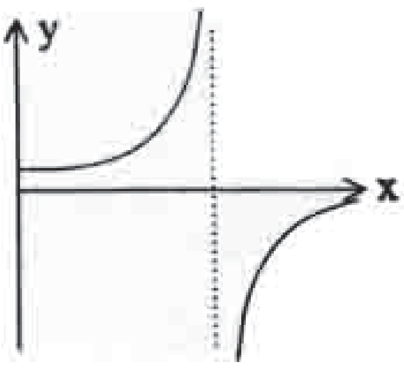


$y = -ab^x$

$y = \frac{-a}{x}$

$y = \ln(ax)$

$y = \frac{-1}{e^x}$

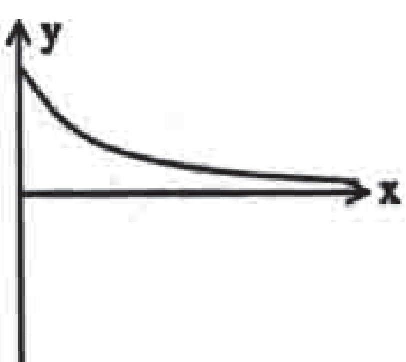


$y = \frac{-a}{x+b}$

$y = \frac{a}{x-b}$

$y = \frac{1}{e^x}$

$y = \frac{-a}{x-b}$



$y = ab^x; \quad b>1$

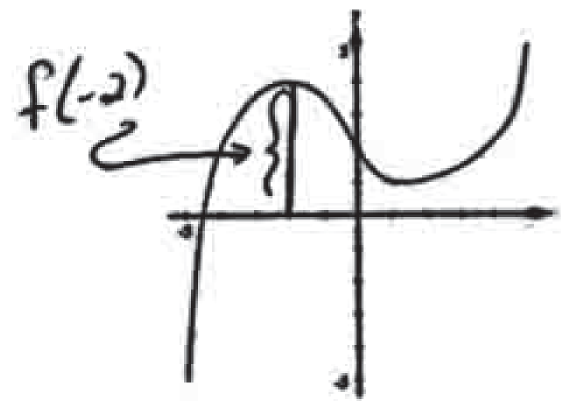
$y = ab^x; \quad 0<b<1$

$y = -\ln(x)$

$y = -e^x$

2. (9 pts.) For each of the graphs of $y = f(x)$ below, show how the indicated quantity can be represented as either the length of a line segment or as the slope of a line. If the quantity is a length, circle the word "length" and clearly draw and label the line segment. If the quantity is a slope, circle "slope" and draw and label the line for which the quantity represents the slope.

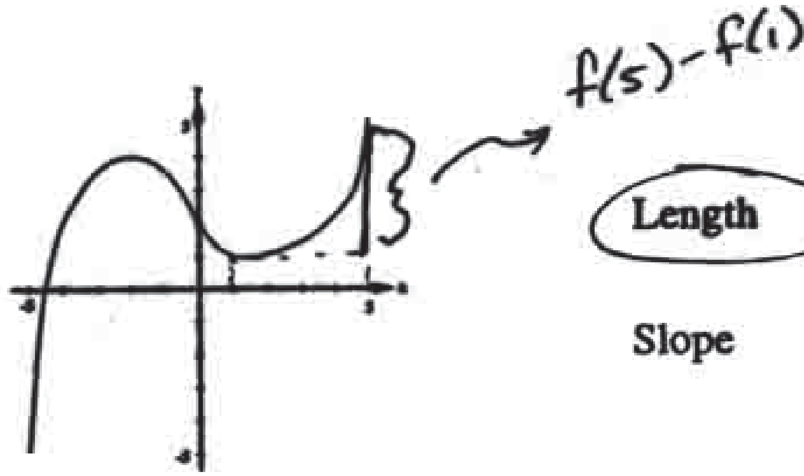
(i) $f(-2)$.



Length

Slope

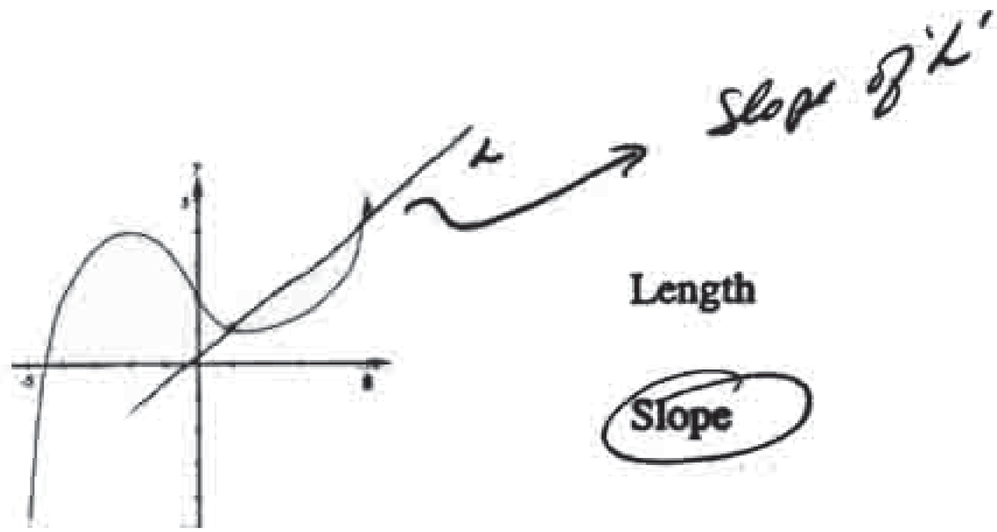
(ii) $f(5) - f(1)$



Length

Slope

(iii) $\frac{f(5)-f(1)}{5-1}$



Length

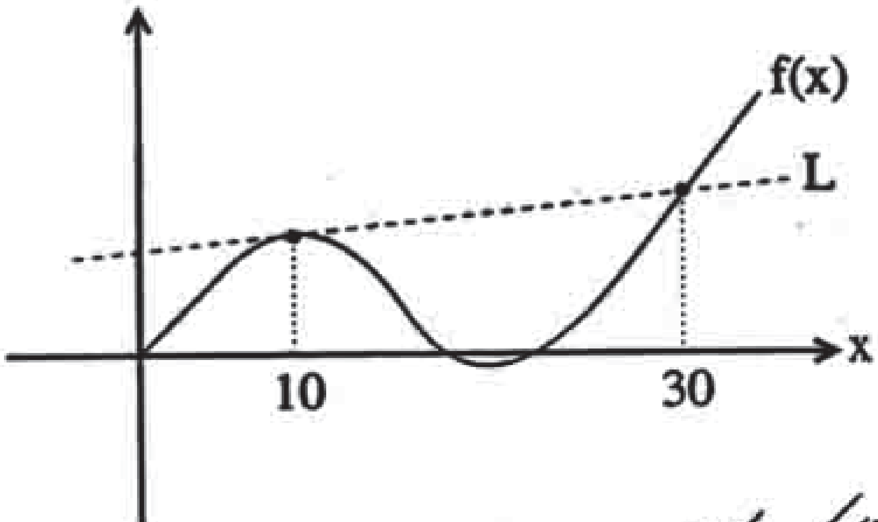
Slope

3. (3 pts.) Let $g(x) = \ln(x^2+3)$. What is the average rate of change in g over the interval from -1 to 3?

Avg Rate of Chn btw $x=3$ & $x=-1$ $= \dfrac{g(3)-g(-1)}{3-(-1)} = \left(\dfrac{\ln(12)-\ln(4)}{4}\right)$

$\approx .27465$

4. (4 pts.) Shown below is a part of the graph of the function f together with a part of the graph of the tangent line L to f at the point $x = 10$. Suppose that $f(10) = 8$ and $f'(10) = 0.12$. Calculate $f(30)$.



pt $(10, 8)$

$m = 0.12$

$8 = 0.12(10) + b$

$b = 6.8$

$f(30) = 0.12(30) + 6.8$

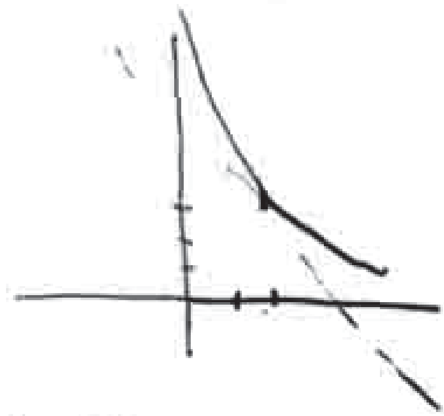
$= 10.4$

$f(30) =$ 10.4.

5. (5 pts.) A function f satisfies the following conditions: $f(2) = 3$, $f'(2) = -2$, and $f''(x) > 0$ for all x.

(a) Circle each of the following numbers that is a possible value for $f(1)$.

1 5 9



(b) Explain the reason for your answers.

Since f is concave up for all x, the graph lies above the tangent line. The point (1,5) is on the tangent line, and (1,1) is below the tangent line. Thus, f(1) = 9 is the only possible choice given.

6. (7 pts.) Ebeneezer borrows \$10,000 to help pay the cost of his college expenses. No interest is charged on the loan while Ebeneezer is in school. After graduation, the loan starts accruing interest at an annual rate of r% per year. He plans to pay off the loan by making equal monthly payments over a 10 year period. Let $C(r)$ denote the total cost of Ebeneezer repaying the loan when the interest rate is r%.

(a) What are the units of $C'(r)$?

The units of C'(r) are dollars per percent or \$/%.

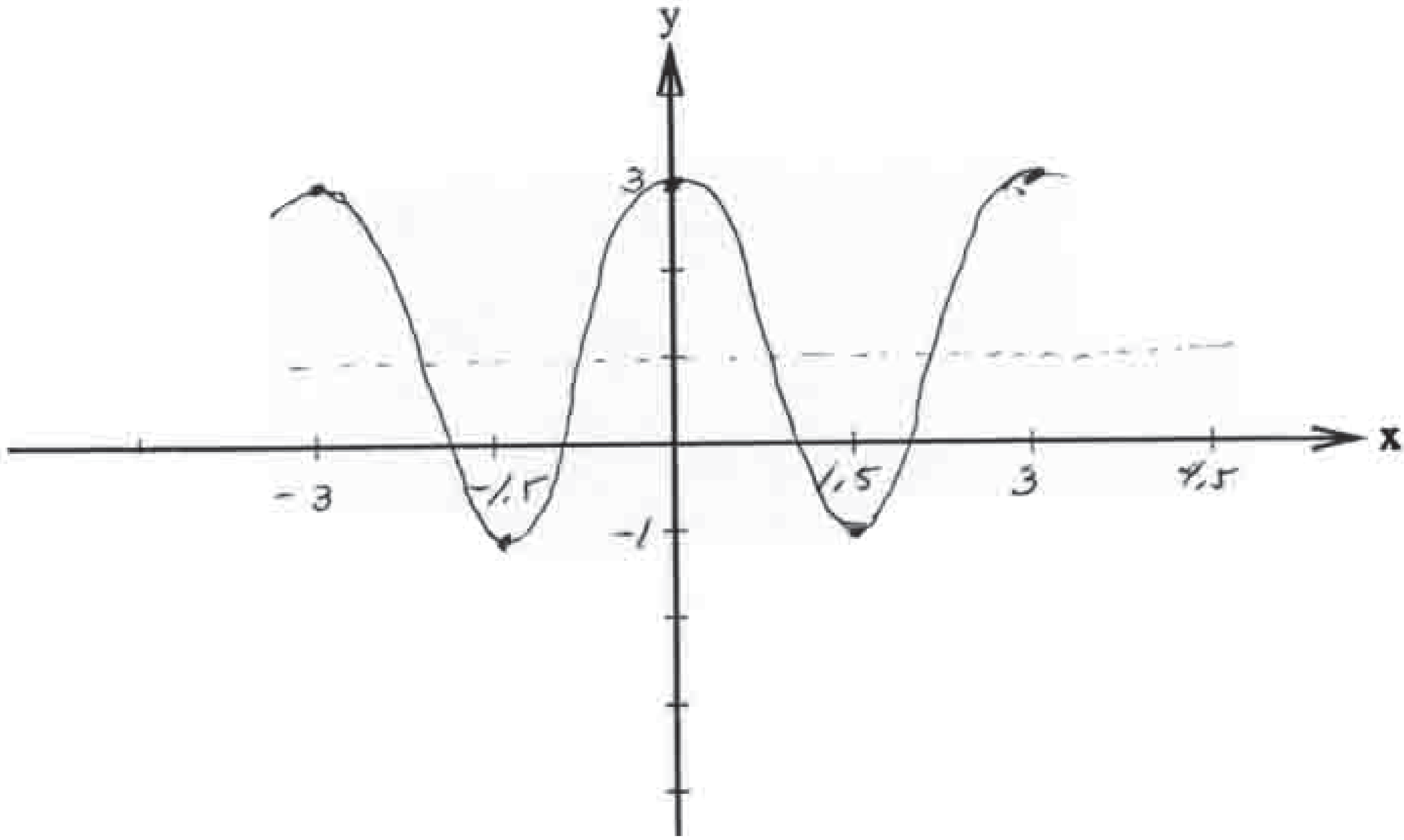
(b) What is the practical meaning of the equation $C'(5) = 586$?

The equation C'(5) = 586 indicates that if the interest rate increased from 5% to 6%, the total cost of repaying the loan would increase by approximately \$586.

7. (10 pts.) On the axes provided below, sketch at least two full periods of the graph of the trigonometric function

$$f(x) = 1 + 2\cos(\frac{2\pi}{3}x).$$

Be sure to indicate the choice of units on each axis.



(b) What are the amplitude and period of f?

Amplitude = 2 Period = 3

(c) Find a formula for the function g whose graph is obtained by shifting the graph of f down by two units and to the right by two units.

$g(x) =$ $-1 + 2\cos\left(\frac{2\pi}{3}(x-2)\right)$

(d) Find a formula for the trigonometric function, k, whose graph has all of the following features

- the same midline and amplitude as f,
- twice as many peaks and valleys as f, and
- at least one of its peaks coincides with a peak of f.

$k(x) =$ $1 + 2\cos\left(\frac{4\pi}{3}x\right)$

8. (7 pts.) A function f is defined for all values of x and the following is a partial table of its values.

x	-0.75	-0.5	-0.25	0.000	0.25	0.5	0.75
$f(x)$	1.968	2.214	2.490	2.800	3.149	3.542	3.983

.246 .276 .31 .349 .373 .441

(a) Using only the function values from the table, give your best estimate for $f'(0)$.

Either: $\frac{3.149-2.800}{.25} = 1.396$

or $\frac{2.8-2.490}{.25} = 1.24$

or $\frac{3.149-2.490}{.5} = \frac{1.396+1.24}{2} = 1.318$

any of these

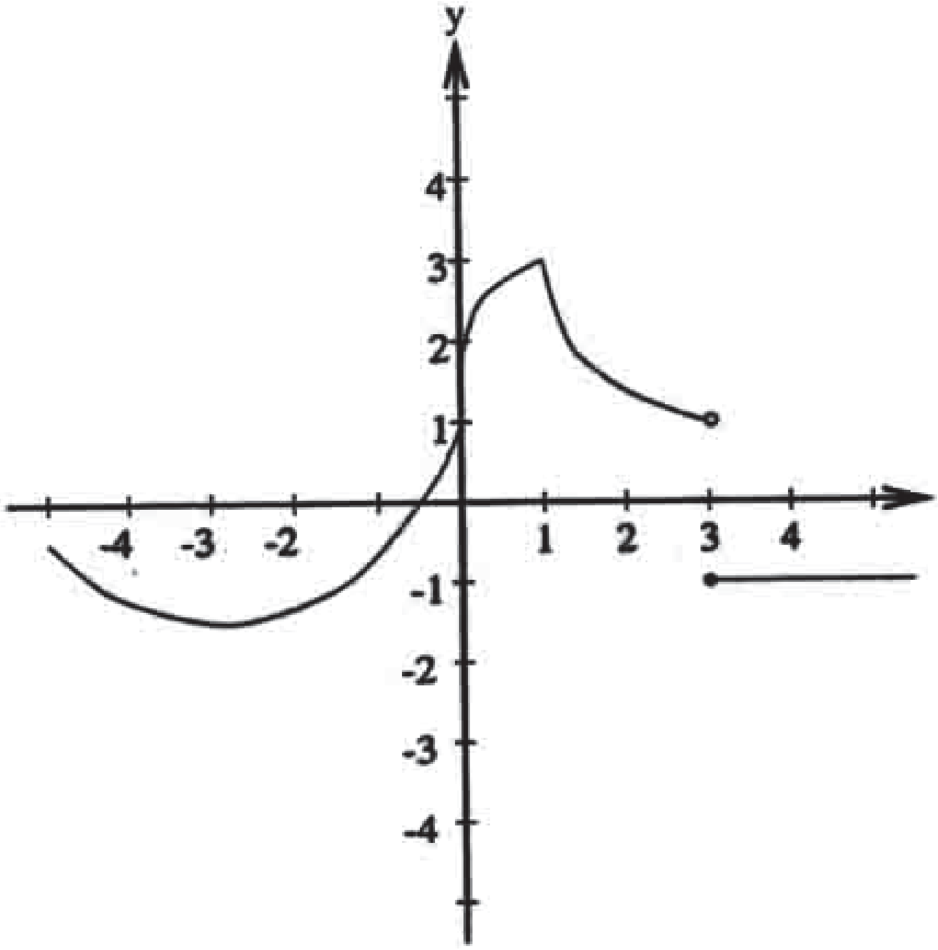
(b) Based on these function values, would you expect that the function f is concave up, concave down, linear, or none of these? Why?

The function appears to be concave up, because the rate of change is increasing. Note that the Δx is .25 for each, and

$\Delta f = .246, .276, .31, \quad , .441.$

Explain your reaso ng clearly.

pts. The graph of the funct f for is hown in the figure below



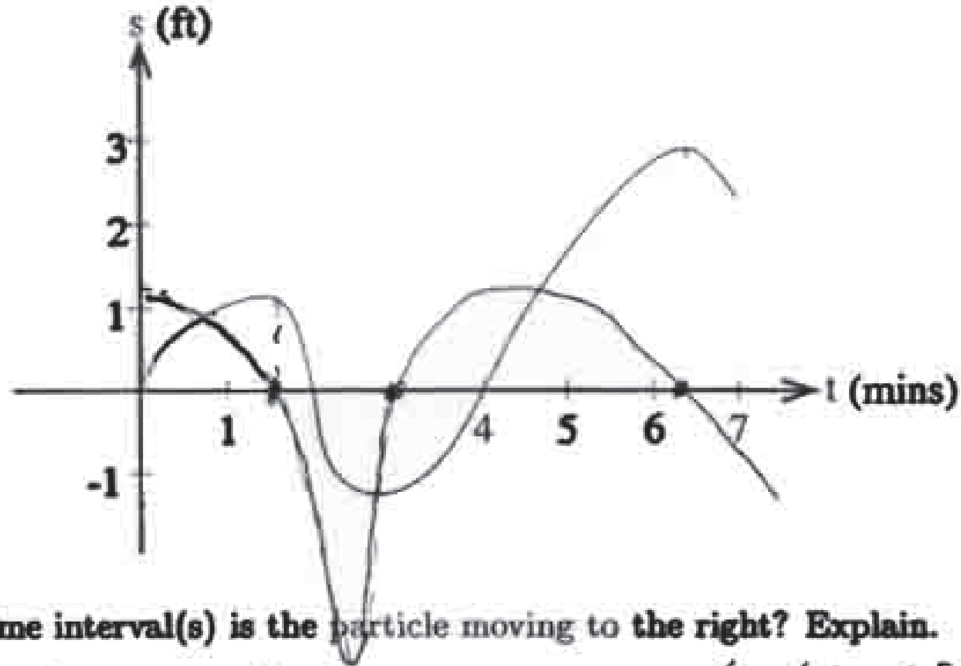
At which point(s) if any does f fail to **be** continuous? Explain why.

At x 3 the function is discontinuous
as there is a break in the graph
(or jump)

(b **A** which point(**s**) if any, does fail to be differentiable Explain why

The function appears to be
non differentiable at
$x = 0$ — vertical tangent
x 1 sharp corner
and x 3 not continuous

10. (14 pts.) A particle is moving along a straight line. Its distance, s, measured in feet to the right of a fixed point at time t minutes, is given by the graph in the figure.



(a) Over which time interval(s) is the particle moving to the right? Explain.

The particle is moving to the right when 'S' is increasing. Thus, approximately for $0 < t < 1.5$ and $3 < t < 6.25$.

(b) Over which time interval(s) does the particle have negative acceleration? Explain.

The particle has negative acceleration when 'S' is concave down, or for approximately $0 < t < 2$ and $4 < t < 7$.

(c) At approximately which time does the particle have the highest speed? (Recall that speed is the magnitude of the velocity.) Explain your answer.

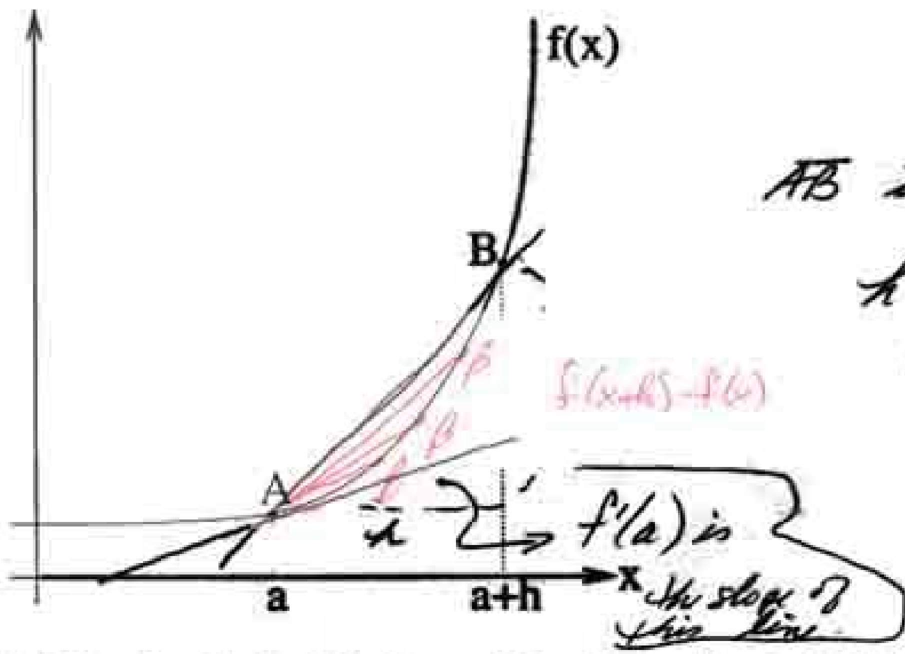
The highest speed is indicated by the steepest slope (in either direction). This appears to be around $t = 2$.

(d) On the axes above, sketch a graph of the velocity function.

11. (11 pts.) (a) Give the limit definition of the derivative of a function f at a point a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) One interpretation of the derivative, $f'(a)$, is that it represents the slope of the tangent line to the graph of f at the point A. Use the limit definition of the derivative and the figure below to show why this interpretation is valid. Feel free to use the space to the right to explain your drawing.



The slope of the line AB is $\frac{f(a+h)-f(a)}{h}$. As $h \to 0$, then $B \to A$ and the slope of AB approaches the slope of the tangent to the curve at A.

(c) Use the limit definition of the derivative to find $g'(x)$ for the function $g(x) = 2x^2 - 3x$.

$$g'(x) = \lim_{h \to 0} \frac{2(x+h)^2 - 3(x+h) - (2x^2 - 3x)}{h}$$

$$\lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) \cancel{-3x} - 3h - 2x^2 \cancel{+3x}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 - 3h \cancel{-2x^2}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 3)}{\cancel{h}} = 4x - 3$$

12. (14 pts.) Sunny and Tyrrell have been dating since New Year's Eve. Sunny has noted that the amount of affection she has for Tyrrell, measured in bushels, is growing at a linear rate. However, since she is a math major, she tells her friend that her affection is growing as the slope of the line tangent to the curve $f(t) = \sqrt{t}$ at the point $(4, 2)$, where t is in weeks since the first of January.

(a) At what rate is Sunny's affection for Tyrrell growing? Write your answer in a complete sentence.

$f(t) = t^{1/2}$ $\quad f'(t) = \frac{1}{2}t^{-1/2}$

$f'(4) = \left(\frac{1}{2\sqrt{4}}\right) = \frac{1}{4}$ Sunny's affection is growing at the rate of $\frac{1}{4}$ bushel per week

(b) Find an equation of the line that is tangent to f at the point $(4,2)$. This is the model for Sunny's affection, $S(t)$.

$S(t) = \frac{1}{4}t + 1$ $\qquad 2 = \frac{1}{4}(4) + b$
$b = 1$

Tyrrell, being an applied mathematician, determines that he can model his affection for Sunny according to the power function $T(t) = kt^2$ (again in terms of bushels and weeks).

(c) If Tyrrell's model passes through the point (8,3), what is k?

$3 = k(8)^2 \longrightarrow k = \frac{3}{64}$

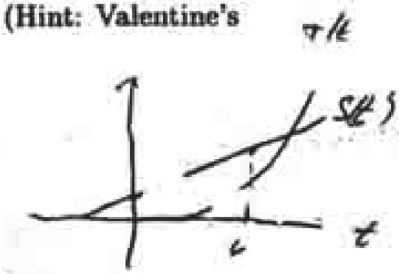
(d) If Sunny and Tyrrell's affection models continue to hold, and if the person with the most affection for the other buys Valentine flowers, who will buy the flowers? Explain. (Hint: Valentine's day is two days from now.)

Valentine's Day occurs @ $t \approx 6$.

$S(6) = \frac{6}{4} + 1 = 2.5$ bushels of affection.

$T(6) = \frac{3}{64}(36) = 1.6875$ bushels of affection.

Sunny will buy the flowers.



(e) Is there a time that Sunny and Tyrrell will have equal affection for one another? If so, approximately when. If not, why not?

If so then

$\frac{1}{4}t + 1 = \frac{3}{64}t^2$

$16t + 64 = 3t^2$

$3t^2 - 16t - 64 = 0$

$(3t+8)(t-8) = 0$

$\cancel{t = -\frac{8}{3}}$ or $t = 8$

discard

Yes, Sunny & Tyrrell will have equal affection for one another at $t = 8$, or in early March. At that time each will have 3 bushels of affection for the other person.