Solution Key

MATH 115 — MIDTERM EXAM II

DEPARTMENT OF MATHEMATICS
University of Michigan

March 26, 2003

NAME: ______________________________

INSTRUCTOR: ______________________ SECTION NO: ________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it.
6. If you use graphs or tables to obtain an answer, be certain to provide an explanation and a sketch of the graph to make it clear how you arrived at your solution.
7. Include units in your answers where appropriate.
8. You are allowed two sides of a 3 by 5 card of notes. You may also use your calculator.
9. Please turn off all cell phones.

PROBLEM	POINTS	SCORE
1	6	
2	11	
3	12	
4	8	
5	8	
6	10	
7	12	
8	12	
9	8	
10	13	
TOTAL	100	

1. (6 points) [Circle the correct answer, no explanation necessary.]
The graph of f is given in the figure below. If f is a polynomial of degree 3, then the values of $f'(0)$, $f''(0)$, and $f'''(0)$ are, respectively,

(a) $0, 0, +$

(b) $0, 0, -$

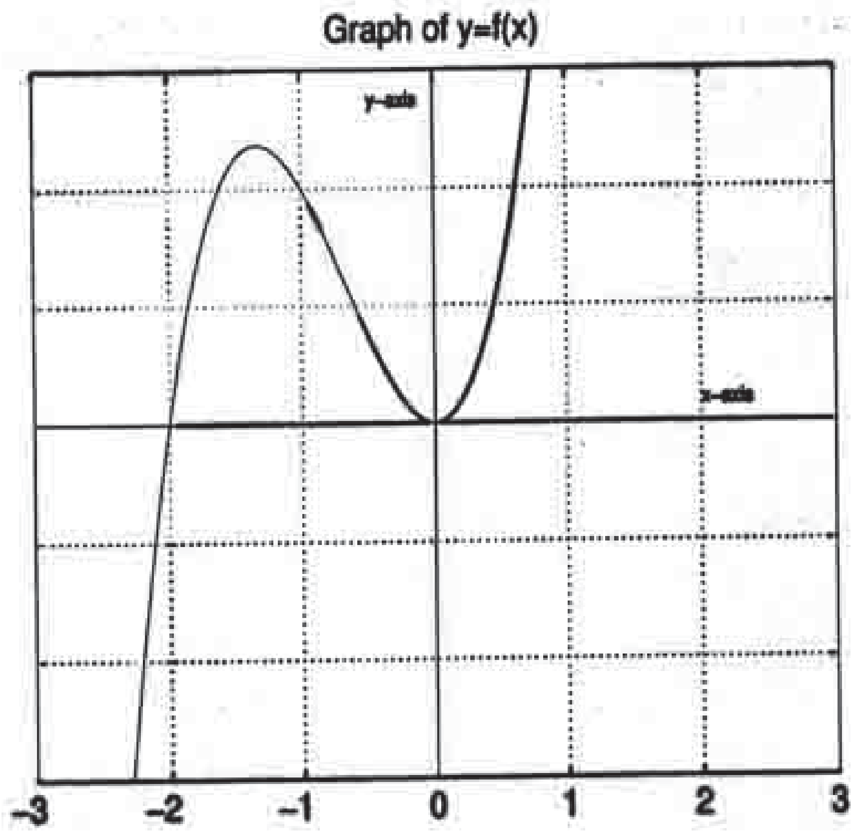
(c) $0, +, -$

(d) $0, -, -$

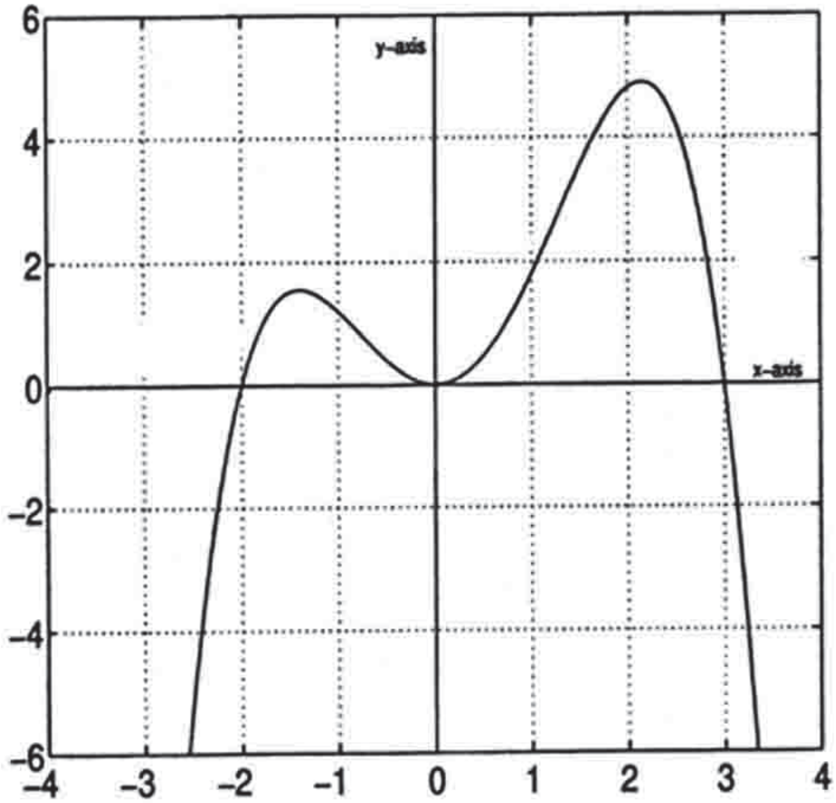
(e) $+, -, +$

(f) $0, +, +$

(g) none of the above



2. (11 points) The graph in the figure below is the graph of f' (i.e., the derivative of the function f).



(a) For what value(s)of x, if any, does f have a critical point? $x = -2, 0, 3$

(b) For what value(s) of x, if any, does f have a local maximum? $x = 3$

(c) For what value(s) of x, if any, does f have a local minimum? $x = -2$

(d) For what value(s) of x, if any, does f have an inflection point? $x = -1.5, 0, 2.1$

(e) Over what intervals, if any, is f increasing? $(-2, 3)$ or $(-2, 0)$ & $(0, 3)$

(f) Over what intervals, if any, is f concave up? $x < -1.5$, $0 < x < 2.1$

3. (12 pts.) Some of the values of the functions f, g, f', g' are given in the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	-1/3	-8/3

(a) If $h(x) = g(f(x))$, then $h'(0) =$ $\underline{-40/3}$.

$$h'(x) = g'(f(x)) \cdot f'(x)$$

$$h'(0) = g'(f(0)) \cdot f'(0)$$

$$= g'(1) \cdot 5$$

$$= \left(-\frac{8}{3}\right)(5) = -\frac{40}{3}$$

(b) If $h(x) = \frac{f(x)}{g(x)+2}$, then $h'(1) =$ $\underline{13/6}$.

$$h'(x) = \frac{f'(x)(g(x)+2) - f(x)g'(x)}{(g(x)+2)^2}$$

$$h'(1) = \frac{\left(-\frac{1}{3}\right)(-4+2) - (3)(-8/3)}{(-4+2)^2} = \frac{(2/3)+8}{4} = \frac{26/3}{4} = \frac{26}{12}$$

$$= \frac{13}{6}$$

(c) If $h(x) = \ln(f(x)$, then $h'(1) =$ $\underline{-1/9}$.

$$h'(x) = \frac{1}{f(x)} \cdot f'(x)$$

$$h'(1) = \frac{1}{3} \cdot (-1/3) = -\frac{1}{9}$$

4. (8 points) A spherical snowball is melting so that its surface area decreases at the constant rate of 40 cm^2 per minute. The surface area and volume of a sphere of radius r are $S = 4\pi r^2$ and $V = 4\pi r^3/3$, respectively. Use this information to answer the following, and remember to include appropriate units in your answers.

(a) How fast is the radius of the snowball changing when the radius is 5 cm?

$S = 4\pi r^2$ given $\frac{dS}{dt} = -40 \frac{cm^2}{min}$

$\frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{1}{8\pi r} \cdot \frac{dS}{dt}$

when $r = 5$, $\frac{dr}{dt} = \frac{1}{8\pi(5)}(-40) = \frac{-1}{\pi} \frac{cm}{min}$

(b) How fast is the volume changing when the radius is 5 cm?

$V = \frac{4\pi}{3} r^3$

$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$, so when $r = 5$ $\frac{dV}{dt} = 4\pi(25) \cdot \left(-\frac{1}{\pi}\right)$

$\left.\frac{dV}{dt}\right|_{r=5} = -100\ cm^3/min$

5. (8 points)

(a) Find the tangent line approximation for $f(x) = \frac{x}{x-1}$ near $x = 3$.

near $x = 3$, $f(x) \approx f(3) + f'(3)(x-3)$

$= \frac{3}{2} - \frac{1}{4}(x-3)$

$f(3) = \frac{3}{2}$

$f'(x) = \frac{(x-1)-x}{(x-1)^2}$

$= \frac{-1}{(x-1)^2}$

$f'(3) = \frac{-1}{4}$

(b) Is the approximation an overestimate or an underestimate of $f(x)$ for values of x near 3? Explain.

underestimate

We have $f'(x) = -(x-1)^{-2}$,

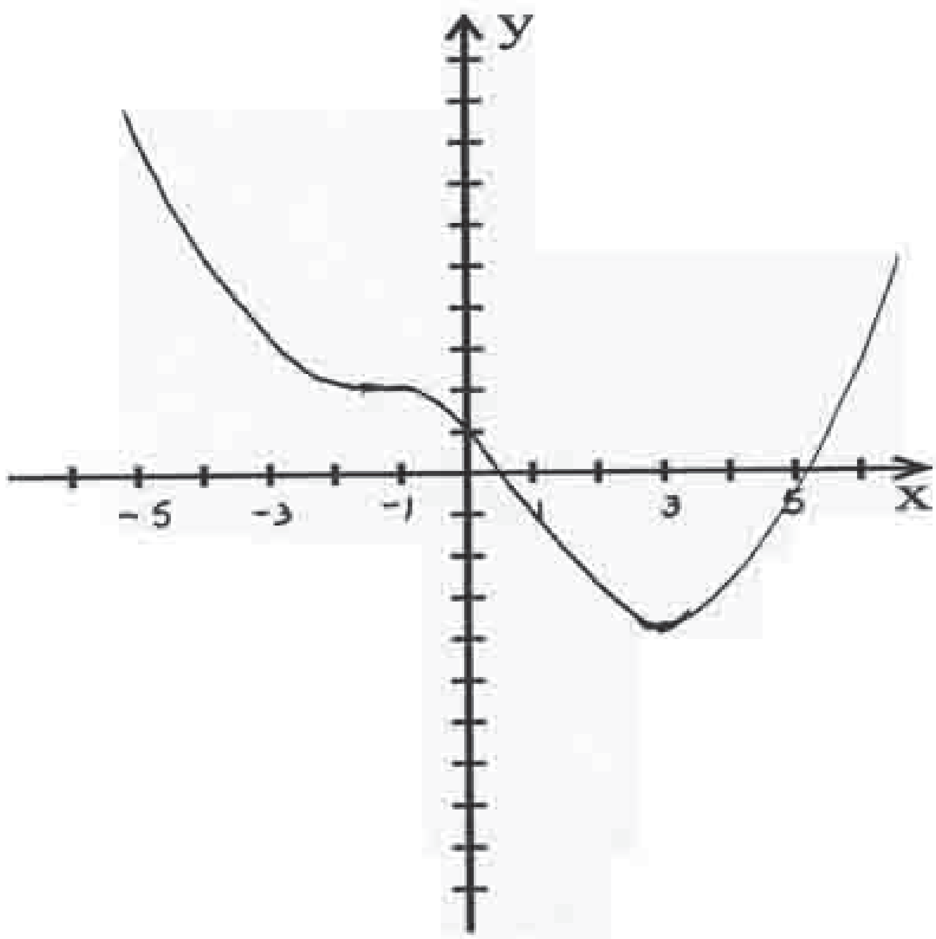
so $f''(x) = 2(x-1)^{-3}$

$f''(3) > 0$ which indicates f is concave up at $x = 3$. In fact, $f''(x) > 0$ for all $x > 1$.

Thus, the approximation near $x = 3$ is an underestimate.

6. (10 points) On the axes below, sketch a possible graph of a single function, $y = f(x)$, given that: [Be sure to show appropriate labels on the x axis.]

- f is defined and continuous for all real x
- f has critical points at $x = -1$ and $x = 3$
- f is decreasing for $x < 3$
- $f'(x) > 0$ for $x > 3$
- f has inflection points at $x = -1$ and $x = 1$
- f'' is positive for $x < -1$



7. (12 points)

(a) Find $\frac{dy}{dx}$ given the equation $y^3 - xy = 2$.

$$3y^2\frac{dy}{dx} - \left(x\frac{dy}{dx} + y\right) = 0$$

$$(3y^2 - x)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

(b) Is there a point, (x_0, y_0), where the tangent to the curve is horizontal (i.e., parallel to the x-axis)? If so, find one. If not, explain why not.

If $\frac{dy}{dx} = 0$, then $y = 0$. However, if $y = 0$, then $0 - x(0) = 2 \rightarrow 0 = 2$. *
Thus, there is no solution for $\frac{dy}{dx} = 0$.

(c) Show that the point (3,2) lies on the curve, and find the equation of the tangent line to the curve at (3,2).

$$(2)^3 - (3)(2) = 8 - 6 = 2 \;\therefore$$ Thus (3,2) [illegible]

$$\left.\frac{dy}{dx}\right|_{(3,2)} = \frac{2}{12-3} = \frac{2}{9}$$

The line is: [illegible]

or $y = \frac{2}{9}x + \frac{4}{3}$

(d) Use local linearization to find a good approximation for a value of y when the point $(3.09, y)$ lies on the curve. [Show your work.]

$$y \approx 2 + \frac{2}{9}(3.09 - 3) = 2 + \frac{2}{9}(.09)$$

$$= 2 + 2(.01) = 2.02$$

8. (12 points) Each day a certain factory in Detroit produces high quality ball-bearings for use in the auto-racing industry. The daily revenue, R, and the daily cost, C, obtained from producing and selling q ball bearings are given by:

$$R(q) = \frac{1}{3}q^3 + 15q, \quad \text{and} \quad C(q) = 4q^2 + \frac{34}{3},$$

where both revenue and cost are measured in tens of thousands of dollars, and the quantity, q is in thousands of ball bearings. The Detroit factory produces a minimum of 1000 bearings a day, but due to machinery and manpower constraints, the factory can produce no more than 6000 ball-bearings in a day. In your answers to the following, be sure to include units where appropriate.

(a) What are the daily fixed costs at the factory?

$\frac{34}{3}(10{,}000)$ dollars $\approx \$113{,}333$.

(b) Recall that the profit function, π, is given by $\pi(q) = R(q) - C(q)$. For which value(s) of q is the daily profit maximized at the Detroit factory? [Show how you determine your answer.]

$\pi(q) = \frac{1}{3}q^3 + 15q - 4q^2 - \frac{34}{3}$

$\pi'(q) = q^2 + 15 - 8q$

$\pi(q) = 0$ if $q^2 - 8q + 15 = 0 \rightarrow (q-3)(q-5) = 0$.

Thus, critical pts @ $q = 3$ & $q = 5$. Endpoints: $q = 1$ & $q = 6$.

$\pi(1) = 0$
$\pi(3) = 6.667$
$\pi(5) = 5.333$
$\pi(6) = 6.6667$

$\rightarrow$ A quantity of 3000 or 6000 ball bearings will maximize the daily profit.

(c) The maximum daily profit at the factory is $66,667.

(d) Another ball-bearing factory is located in Pittsburgh. The daily revenue and cost functions of this factory are identical to the ones above. However, the Pittsburgh factory produces between 500 and 8000 ball-bearings a day (depending on who is playing in Heinz Field). For which value(s) of q is the daily profit maximized at the Pittsburgh factory, and what is the maximum profit?

Note: the cp's $q = 3$ & $q = 5$ are the same.
The endpoints are $q = \frac{1}{2}$ & $q = 8$.

$\pi(.5) = -4.79$
$\pi(8) = 23.33\overline{3}$

The Pittsburgh factory's profit is maximized when they produce 8000 ball-bearings. Max profit is $233,333

9.(8 points)

(a) Show that the function $f(x) = e^{-x^2/2}$ is concave down on the interval $-1 < x < 1$ and concave up if $x > 1$ or $x < -1$. [Be sure to show your work.]

$$f'(x) = -x\left(e^{-x^2/2}\right)$$

$$f''(x) = (-1)\left(e^{-x^2/2}\right) + (-x)\left(-xe^{-x^2/2}\right)$$

$$= e^{-x^2/2}\left(x^2-1\right) = e^{-x^2/2}(x+1)(x-1)$$

$(x-1)$ - - - - - - - - - - 0 + + + +

$(x+1)$ - - - 0 + + + + + + +

f'' + -1 $-$ 1 +

Note: $e^{-x^2/2} > 0$ for all x.

$(x-1)(x+1) > 0$ for $x < -1$ or $x > 1 \rightarrow f$ is concave up.

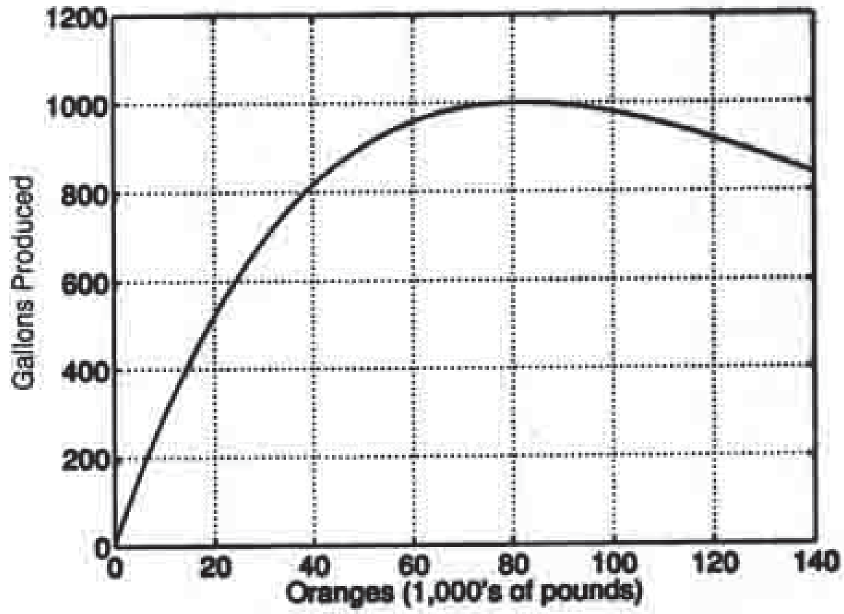
$(x-1)(x+1) < 0$ for $-1 < x < 1 \rightarrow f$ is concave down

(b) Find the member of the family of functions given by $y = e^{-(x-a)^2/b}$ that has a maximum at $x = 3$ and is concave down on the interval $1 < x < 5$.

The function $f(x) = e^{-x^2/2}$ has a max @ $x = 0$. We want the max @ $x = 3$, so we need to shift the graph right by 3 units. Thus, $a = 3$. We also want a horizonal stretch from the max by a factor of 2, so we divide "on the inside" by 2. Thus,

$$\boxed{y = e^{\frac{-(x-3)^2}{8}}}$$

10. (13 points) To qualify as "fresh squeezed," orange juice must be made from oranges which are juiced the same day that they are picked. The manager of the juicing factory attached to the orchard, Mr. I. Squeezem, knows that his factory can handle up to 120,000 pounds of oranges per day. However, due to inefficiencies in storage and complications of the production process when busy, the factory produces only $g(z)$ gallons of orange juice in a day when supplied with z thousand pounds of oranges, where g has the graph given in the figure.



At the orange orchard the harvesting season lasts for 60 days and on day t of the harvesting season

$$f(t) = 80\sin\left(\frac{\pi t}{60}\right)$$

thousand pounds of oranges are harvested and delivered to the juicing factory.

(a) Mr. Squeezem passed math 115 many years ago. In terms of oranges harvested, explain the meaning of the statement "$f'(20) \approx 2.09$" to him.

Twenty days into the harvest season, the delivery of oranges to the factory is increasing at the rate of approximately 2,090 lbs per day.

(b) On which day(s) of the harvesting season is the greatest weight of oranges delivered to the factory?

$f'(t) = \frac{80\pi}{60}\left(\cos\frac{\pi t}{60}\right)$ day 30

On the interval $(0, 60)$, $f'(t) = 0$ when $t = 30$. Looking at the graph of $f(t)$ on $[0, 60]$, we see that $t = 30$ is a max + since that is the only c.p. it is the global max.

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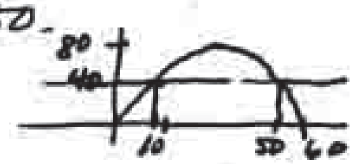
(c) On which day(s) of the harvesting season are 40,000 pounds of oranges delivered?

From the graph, we see that 40,000 lbs are delivered on days 10 & 50.



(d) From his experience at the factory, Mr. Squeezem knows that when 40,000 pounds of oranges are at the factory, he can produce an additional 10.4 gallons of orange juice from an additional 1000 pounds of oranges. Show Mr. Squeezem how he could express this very simply in terms of the function g if he remembered his calculus.

$$g'(40) = 10.4$$

(e) On the day(s) that 40,000 pounds of oranges are delivered to the factory, at what rate is the number of gallons of orange juice produced changing per day. [Show clearly how you obtain your answer.]

Note that $z = f(t)$, so

$g(z) = g(f(t))$ and

$\frac{dg}{dt} = g'(f(t)) \cdot f'(t)$ so

Note: when $f(t)=40$, $t=10$ or 50.

Thus, $\left.\frac{dg}{dt}\right|_{\substack{z=40 \\ \to t=10}} = g'(40) \cdot f'(10)$

$f'(t) = \frac{8\pi}{6}\left(\cos\frac{\pi t}{60}\right)$

$f'(10) = \frac{8\pi}{6}\left(\cos\frac{\pi}{6}\right)$

$= \frac{4}{3}\pi\left(\frac{\sqrt{3}}{2}\right)$

≈ 3.6276

$f'(50) \approx -3.6276$

Therefore, $\left.\frac{dg}{dt}\right|_{t=10} = 10.4(3.6276)$

$\approx 37.727.$

The rate is increasing at the rate of approximately 37.73 gallons/day on day 10 and decreasing by 37.73 gal/day on day 50.